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Abstract : Production yield can be highly volatile and uncertain, especially in industries where exogenous and environmental factors such as the climate or raw material quality can impact the manufacturing process. To address this issue, we propose a multiperiod single-item lot-sizing problem with backorder under yield uncertainty via a robust optimization methodology. First, we formulate the robust model based on the budgeted uncertainty set. The resulting model optimizes under the worst-case perspective to ensure the feasibility of the proposed plan for any realization of the uncertain yield. Second, we derive the properties of the optimal robust policies for the special cases under a box uncertainty, which helps us obtain a dynamic program with polynomial complexity. Finally, extensive computational experiments show the robustness and effectiveness of the proposed model through an average and worst-case analysis. The results demonstrate that the robust approach immunizes the system against uncertainty. In addition, a comparison with the stochastic models shows that the robust model balances the costs to reduce the backorders at the expense of more often producing a larger amount of goods.

Keywords: Lot-sizing, production, uncertainty, robust optimization

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1 Introduction

Many industries constantly face exogenous factors that can affect the quality of their products. In addition, recent products have become increasingly more complex with shorter life cycles, and product customization breaks the regularity of the production process and increases the number of failure sources [1]. It becomes particularly more challenging to precisely estimate the production yields that are necessary in the production planning process. In this context, accounting for this yield uncertainty during production planning is crucial because an underestimation of the production yield leads to excess inventory, whereas an overestimation creates significant stock-outs [2, 3].

The production yield rate incorporates quality factors into the lot-sizing model by measuring the expected quantity of non defective items resulting from the release of a given production lot. Traditionally, this production yield rate is estimated based on historical data or machine specifications, but such estimations can be imprecise. In practice, this rate is subject to multiple sources of uncertainty, such as irregularities in the raw materials, deviations from standard operating procedures, failures in the machinery of the system, inefficiency, a lack of control, or a lack of a quality control system. Yield uncertainty issues occur in many industries, such as electronic [4, 5, 6], pharmaceutical [7], food [8], agricultural [9, 10, 11], steel and metallurgical industries [12], as well as in remanufacturing processes [13].

Lot-size decisions are a crucial step in production planning when aiming to meet customers' needs and minimize the overall costs [14]. Although modelers often rely on deterministic variants of lot-sizing models based on the hypothesis that all data are known or can be correctly predicted, in practice, many parameters are uncertain [15]. The discrepancy between the estimated data and their actual values can have a critical impact on the quality of the lot-sizing decision. Therefore, there is a need for developing lot-sizing methods that account for these uncertainties, the current work investigates the use of robust optimization for lot-sizing under yield uncertainty.

The yield uncertainty may have different impacts depending on the situation, such as an increase in the production costs, processing duration, or lead times, and it often results in wasting of materials and the available resources. The consequences of these losses can be highly damaging to the system [16]. Most studies on lot-sizing problems (LSPs) with uncertain yield consider the single-item single-period problem [2, 3]. In this simple setting, the optimal production quantity can be derived through a mathematical analysis based on the newsboy inventory management model [3]. This technique, however, cannot be applied in a more general context of multiperiod lot-sizing, and it can lead to poor solutions [2].

Stochastic and robust optimization are promising alternative methodologies because they usually account for uncertainty in the parameters of the models. The present paper proposes a methodology based on robust optimization for the non stationary multiperiod LSP under yield uncertainty, and we analyze it in terms of its applicability, optimality, and efficiency. To the best of our knowledge, we are the first to consider robust optimization for a LSP under yield uncertainty in a dynamic production context, where the production parameters such as costs, demands, and production yield rates may change at each production period. In a similar vein, [17] propose a stochastic optimization approach to deal with LSPs under yield uncertainty. The authors show that the stochastic method is efficient within a static strategy because it minimizes the occurrence of the backorder. Considering the robust optimization approach, [18] tackle the stationary inventory management model with an uncertain production yield and fixed inventory and backordering costs, but they ignore the setup decisions and production costs. The authors propose a constraint sampling approximation to mitigate over conservative solutions. In a similar spirit, [19] propose a robust model for the inventory management problem with demand and yield uncertainties; they consider a stationary case where the costs and additional parameters are fixed for the entire production horizon. They show that the problem can be formulated as a nominal problem with modified deterministic demand in terms of the accumulated deviation of both the uncertain demand and uncertain yield. Even though [19] propose an insightful analysis of the inventory management problem with uncertain yield, they perform their studies in a procurement perspective, for which the production yield rate is set to its nominal value. Thus, [19] do not allow for obtaining an amount of quality goods larger than the nominal ordered quantities. Hence, we note a lack of studies on the application of the robust optimization methodology to the LSPs with yield uncertainty.

The current paper aims to fill the knowledge gap on the impact of the uncertain production yield on a multiperiod lot-sizing problem within a robust production planning perspective for a non stationary case of costs and demands. The contribution of the current paper is fourfold. First, we propose a robust optimization formulation for a multiperiod LSP under yield uncertainty for the non stationary case of costs and demands under the budgeted uncertainty set. Second, we derive an optimal policy for the stationary case of the nominal and maximum deviation values of the uncertain yield. This special case considers the box uncertainty set without setup and production costs, much like the case for the demand uncertainty presented by [20]. Third, we propose a polynomial-time dynamic programming algorithm for a special case that can be extended to the version with a stationary setup cost. Finally, we perform an in-depth analysis of the resulting methods in terms of the quality of the solution, scalability, stability, robustness, and flexibility. In particular, we compare the production plans resulting from the robust and stochastic models. Although the robust models aim to ensure the robustness and feasibility of the proposed plan, the stochastic programs seek the production plan with the best expected costs. Thus, we intend to analyze when each technique is best suited to deal with production yield uncertainty.

The current paper is organized as follows: Section 2 gives a review of previous work on nondeterministic lot-sizing problems, here with a focus on uncertain production yields. Section 3 formally describes the considered problem and introduces the robust optimization methodology. It also presents some properties of the optimal robust policy and a dynamic programming formulation for the special case described above. Section 4 presents the instances and simulation framework used in our experiments, as well as the experimental results. Finally, Section 5 concludes this work and provides some future research directions.

2 Literature review

Because of its practical importance, LSPs has attracted a wide range of research from the manufacturing and mathematical optimization communities. Several authors have suggested exact solution methods [21, 22, 23] and strong formulations [24, 25] for an uncapacitated and capacitated LSP. [26] provide a complete study about mixed-integer programming (MIP) for an LSP, and [27] present a state-of-the-art of the single-item LSP with a focus on the modeling and resolution methods. Although most studies concern a deterministic LSP, there is a growing amount of research on nondeterministic lot-sizing. This section reviews the general studies on the stochastic LSP before focusing on a LSP under yield uncertainty.

2.1 LSPs under uncertainty

The bibliography on LSPs under uncertainty, which is presented by [28], confirms the prevalence of studies on demand uncertainty because this is the most natural source of uncertainty within a production planning context. However, the production yield, lead time, capacity, and cost uncertainty may similarly affect the quality of the solution. The authors indicate that the main optimization approaches that handle a nondeterministic LSP over the last two decades are "mathematical analytical methods, stochastic mathematical programming, deterministic mathematical programming with simulated data, scenario-based robust optimization, fuzzy modeling, stochastic and discrete event simulation, on-line decision mechanisms, meta heuristics, queuing theory, game theory, and interval arithmetic techniques." [28, p. 2294]

A simple approach to handle nondeterministic LSPs founds in its formulation within classical inventory management models. For instance, the newsboy model [3] minimizes the expected costs of overestimating or underestimating the uncertain parameters within a single-period analysis. Although this approach leads to a simple formula to compute the production quantity, it requires strong assumptions, such as single-period and constant parameter values over time. Because these assumptions are often not consistent with real applications, the extensions of the mixed integer linear programming (MILP) formulations to deal with uncertain parameters have emerged as promising methods to improve the quality of the decisions. Among these methods, stochastic programming and robust optimization have stood out for the past few decades [27].

Stochastic programming (SP) [29] represents uncertain parameters with their probability distributions, and the aim here is to make decisions by minimizing the expected costs. Although this methodology is efficient for small-size instances with a limited number of scenarios, it does not scale well for large instances or for a large number of scenarios, notably those within dynamic contexts [30]. For LSPs with an uncertain yield in a static strategy, the stochastic model is still scalable, but it requires the use of a sufficiently large scenario set representing the underlying distributions. Unlike the SP method, robust optimization (RO) does not rely on a probabilistic distribution. RO is another commonly adopted approach for optimization under uncertainty [31]. RO describes the uncertainty with a set of possible values for the uncertain parameters, and it optimizes against the worst-case realization within this set [32]. Although simple RO models are easy to solve in general, they can lead to extremely conservative solutions [31]. We refer the readers to [29] and [33] for an exhaustive state-of-the-art look at stochastic programming and robust optimization, respectively.

2.2 Lot-sizing problems under yield uncertainty

The yield uncertainty concerns the inability to precisely predict the output quantities associated with a production lot size. This uncertainty has diverse causes, such as process dysfunction, material imperfections, capacity limitation, and environmental factors such as temperature and humidity [34]. The production yield rate is a common assumption when modeling the yield uncertainty. This parameter was first introduced by [35], and it was initially measured as the proportion of items that were accepted and that reached a high enough quality to be sent to costumers. We use the same notion to capture the proportion of good quality item in a lot. We refer the reader for more on an LSP under yield uncertainty to an excellent survey presented by [2].

As mentioned by [3], most of the works dealing with LSPs under yield uncertainty consider single-period problems [36, 37, 38, 39]. There are some notable studies that have attempted to tackle multiperiod problems. [40] presents the dynamic program for the newsboy for a multiperiod single-item model. [41] extend the model presented by [40] for a LSP under yield uncertainty with a tardiness cost. In addition, the authors propose a distribution-free approach to obtain optimal single-period and multiperiod policies based on the mean and variance of the yield rate and without assuming the distribution of the uncertain parameter. [42] present an infinite-horizon inventory control model with deterministic demand, here facing the supply chain risk in disruptions and yield uncertainty. The authors solve the model via the approach based on Leibniz's rule. To deal with the multiperiod LSP problems in an uncertain environment, many researchers account for the demand uncertainty and its extensions in the continuous-time generalization of the single-period newsboy problem [34, 37, 43, 44]. In this context, the problem is characterized by several newsboy problems organized and controlled in sequential stages.

To the best of our knowledge, [17]'s study is the only work on stochastic programming for a LSP under yield uncertainty; there exists no study on robust optimization for lot-sizing under yield uncertainty in a production planning context. [18] illustrate the constraint sampling approximation for convex multi stage robust optimization on an inventory management problem under yield uncertainty. Nevertheless, the contribution of their work is limited to simple inventory management without the setup and backorder costs. [19]'s study is the only one on a robust perspective, where the authors consider a procurement perspective of the stationary inventory management problem; they restrict the maximum value of the production yield to its nominal value, and the authors compare the performance of a robust model with demand uncertainty to a robust model with demand and yield uncertainties. They analyze the impact of the budget by controlling the uncertainty and average and standard deviation of the uncertain parameters on the average performance of the robust models. However, they only consider the stationary case, where the cost and additional parameters are fixed for the entire production horizon. In addition, contrary to our study, [19] do not compare the performance of the robust optimization approach against the stochastic programming and deterministic models.

Our work differs from the aforementioned literature in several aspects. First, to the best of our knowledge, the current paper is the first to formulate the non stationary case of single-item and multiperiod LSPs under yield uncertainty via robust optimization. We formulate the problem based on the worst-case methodology presented in [31], duality proof by [45], and the reformulation per constraints technique via a budgeted uncertainty set introduced by [46] and [47]. Second, we derive the optimal robust policies for the stationary single-period and multiperiod LSPs with uncertain yield, zero setup, and zero production costs. Then, we propose a polynomial-time dynamic programming algorithm based on optimal robust policies to solve the considered problem for the special case when setup costs are allowed. Fourth, we provide an in-depth analysis of the impact of robust optimization for production planning based on numerical experiments. The results show that robust optimization is highly efficient and produces production plans that are more robust to different yield scenarios when compared with the stochastic programming approach.

3 Robust Optimization

The RO methodology immunizes the production plan from uncertainties by computing a solution that remains feasible for any realization of the uncertainty within the uncertainty set [32]. In a constraint-wise RO, the uncertain parameters appear only in the constraints of the problem, and they are described through an uncertainty set. A common approach in RO is to reformulate the problem as a robust counterpart (RC) model that is given by the worst-case perspective of the uncertain parameters into a tractable form and then to optimize the reformulated model. A duality approach is often exploited in the reformulation because the best dual on robust models is equal to the worst primal [45].

3.1 Nominal problem

Before introducing the RO methodology for a LSP under yield uncertainty, we present the nominal formulation of the problem. A single-item multiperiod uncapacitated LSP with backordering and production yield rate determines the quantity to produce in each period of the finite planning horizon $T = \{1, ..., |T|\}$. The objective is to meet demands efficiently and with quality goods while minimizing the overall costs. For each period $t \in T$, we are given the setup cost s_t, the unit production cost v_t, the inventory holding cost h_t, the backorder cost b_t, and the demand d_t. The model comprises the following decision variables: the lot size X_t to produce, the inventory level I_t and the backordered level B_t at the end of the period, and the setup decision Y_t, such that $Y_t = 1$ if a setup occurs in t ($X_t > 0$) and $Y_t = 0$ otherwise. We define ρ_t as the production yield rate in period t, with $\rho_t \in [0, 1]$.

The formulation of the deterministic LSP with production yield rate is as follows:

$$\min \quad \sum_{t \in T} s_t Y_t + v_t X_t + h_t I_t + b_t B_t \tag{1}$$

$$s.t. :$$

$$I_t - B_t = I_{t-1} - B_{t-1} + \rho_t X_t - d_t \qquad t \in T \tag{2}$$

$$X_t \leq M Y_t \qquad t \in T \tag{3}$$

$$X_t, I_t, B_t \geq \ 0 \qquad t \in T$$
$$Y_t \in \{0,1\} \qquad t \in T$$

Without a loss of generality, we assume that there is no stock or backorder at the beginning of the planning horizon. The objective function (1) minimizes the total cost comprising the setup, unit production, inventory, and backorder costs. The inventory balance constraints (2) compute the level of backorder and inventory in period t from the demand, the produced goods at period t, and the inventory and backorder levels in period $t-1$. The constraints (3) are setup-forcing constraints that relate the production quantities (X_t) to the setup decisions (Y_t), whereas $M=\frac{\sum_{t\in T} d_t}{\min_{t\in T}\bar{\rho}_t}$. These constraints set the setup variable Y_t to 1 if any production occurs in period t, and the setup remains inactive otherwise ($Y_t = 0$). In addition, constraints (3) can represent the capacity constraint by setting $M=\min\{C,\frac{\sum_{t\in T} d_t}{\min_{t\in T}\bar{\rho}_t}\}$.

3.2 Definition of the uncertainty set

The uncertainty set must yield a tractable form. Modelers often rely on some statistical consideration of historical data or previous knowledge about the studied system. The first robust optimization models used the *box uncertainty set*, introduced by [48], which describes the uncertainty within an interval of possible values and that is bounded by its minimal and maximal acceptable realizations. Because the box uncertainty set leads to an overly conservative solution, [46] propose the *box polyhedral uncertainty set*, which is also know as the budgeted uncertainty set, where the uncertain parameter takes values within a range of values whose size is controlled by the decision-maker through a budget of uncertainty Γ. This Γ reflects the degree of risk aversion of the decision-maker, giving a threshold for the number of uncertain parameters that can take their worst value [49]. This budget is a degree of acceptable variance of the uncertainty from its nominal value, where the larger the budget, the more averse to risk the decision-maker is. As a matter of fact, the box uncertainty corresponds to the polyhedral uncertainty set with a large enough budget Γ. [50] summarize the commonly used uncertainty sets that describe all possible values for the uncertain parameter and do so using only a few parameters. These sets should be understandable by the user, and the resulting model should be tractable, even though the model accounts for an infinite number of realizations of the uncertain parameter.

In the present work, we consider the widely adopted budgeted uncertainty set $\mathcal{U} = \{\mathbf{Z} \in \mathbb{R}^T : \sum_{\tau=1}^{t} |Z_\tau| \leq \Gamma_t \quad t \in \mathcal{T}\}$. This set is based on the nominal value and maximum deviation of the uncertain yield because these values are largely used in statistical quality control to bound the quality in terms of the key performance indicators [51]. These values are easily obtained from historical data, and they reflect the basic characteristics of the uncertain parameter. Thus, we estimate the uncertain yield rate $\widetilde{\rho}_t$ through a natural parameterization $\widetilde{\rho}(Z) = \bar{\rho} + Z\hat{\rho}$, with $Z \in [-1, 1]$. Here, the uncertain yield belongs to a range centered on its nominal value $\bar{\rho}$ and spread by its maximum deviation $\hat{\rho}$. The disturbance arising from the nominal value is given by the term Z. Therefore, we replace the production yield rate ρ_t in constraints (2) by the uncertain production yield $\widetilde{\rho}(Z)$.

3.3 Definition of the approach to solve the robust counterpart

The reformulation of the nominal problem to account for an uncertain parameter leads to a robust counterpart that considers the uncertainty set in a tractable way. This reformulation expresses all the realizations of the uncertain parameter with a finite number of inequalities. There exist two common approaches to handle a robust counterpart [33]: the reformulation per constraint and dualization and the adversarial method. Because the latter is computationally intensive, we consider the reformulation per constraint and dualization approach. To avoid the inclusion of all possible quantifiers of the uncertainty in the uncertainty set, this approach consists of three steps: (1) reformulation of the constraints subject to the uncertainty as a worst-case reformulation; (2) dualization of the reformulation; and (3) dropping the dualized reformulation into the initial formulation without the inner optimization term [31].

3.4 A robust LSP with an uncertain yield

The robust model determines a production plan that minimizes the total costs under the worst realization of the production yield rates within the uncertainty set $\mathcal{U}$. Consequently, the resulting plan is robust against any realization of the production yield rate in the set $\mathcal{U}$.

The robust LSP model with uncertain yield is given as follows:

$$
\begin{aligned}
\min \quad & \sum_{t\in T} s_t Y_t + v_t X_t + H_t && && (4)\\
& s.t.: \\
& H_t \geq \max_{\widetilde{\rho}\in\mathcal{U}} \left[h_t \sum_{\tau=1}^{t} (\widetilde{\rho}_\tau X_\tau - d_\tau) \right] && t\in T && (5)\\
& H_t \geq \max_{\widetilde{\rho}\in\mathcal{U}} \left[-b_t \sum_{\tau=1}^{t} (\widetilde{\rho}_\tau X_\tau - d_\tau) \right] && t\in T && (6)\\
& X_t \leq M Y_t && t\in T \\
& X_t, H_t \geq 0 && t\in T \\
& Y_t \in \{0,1\} && t\in T
\end{aligned}
$$

where the upper bound on the production quantity M can be set based on the lowest possible value of the production yield, that is, $\min_{t\in T}(\bar{\rho}_t - \hat{\rho}_t)$. Thus, for a LSP under yield uncertainty, M is set to $M = \frac{\sum_{t\in T} d_t}{\min_{t\in T}(\bar{\rho}_t - \hat{\rho}_t)}$.

This robust model is similar to the nominal model, but constraints (5) and (6) replace constraints (2). Here, we reformulate the equality constraints by a pair of inequalities based on the convexity and piecewise linearity of the inventory and backorder cost functions. Because the backorder corresponds to a negative stock level [47], these costs are complementary. Therefore, we can drop the inventory and backorder variables, so we directly compute the inventory and backordering costs according to the difference between the number of quality goods and demand. Thus, H_t represents either the inventory or backorder cost in period t, and constraints (5) (resp. (6)) compute the inventory (resp. backorder) costs.

The inventory and backordering cost constraints account for the uncertain yield, and they may be addressed via the reformulation per constraint and dualization approach. For the sake of clarity, we show the detailed steps of reformulation per constraints only for inventory inequalities (5) because the application for backorder inequalities is analogous. In fact, the inequalities differ only by the sign and the costs associated with the inventory level. Thus, the inventory level is negative in case of backorder and positive if the production exceeds the demand. As a result, within the worst-case perspective of the robust approach, the optimal plan corresponds to a decision leading to a higher cost among these two groups of constraints.

The first step of the reformulation approach is the worst-case reformulation given by the following:

$$
H_t \geq h_t \left[\sum_{\tau=1}^{t} (\bar{\rho}_\tau X_\tau - d_\tau) + \max_{\mathbf{Z}\in\mathcal{U}} \sum_{\tau=1}^{t} \hat{\rho}_\tau X_\tau Z_\tau \right] \qquad \forall t\in T. \tag{7}
$$

For the inventory cost constraints, the worst-case scenario occurs only when the deviation is positive, that is, when $\mathbf{Z} \geq 0$. Assuming that λ and μ are the dual variables, we can finally replace the worst-case reformulation with its dual formulation. Therefore, constraints (7) are reformulated as follows:

$$
H_t \geq h_t \left[\sum_{\tau=1}^{t} (\bar{\rho}_\tau X_\tau - d_\tau) + \min_{\lambda_t + \mu_\tau^t \geq \hat{\rho}_\tau X_\tau} \Gamma_t \lambda_t + \sum_{\tau=1}^{t} \mu_\tau^t \right] \qquad \forall t\in T.
$$

Similarly, the application of the reformulation per constraints and the dualization approach for constraints (6) leads to the following reformulation:

$$H_t \geq -b_t \left[\sum_{\tau=1}^{t} (\bar{\rho}_\tau X_\tau - d_\tau) - \min_{\lambda_t + \mu_\tau^t \geq \hat{\rho}_\tau X_\tau} \Gamma_t \lambda_t + \sum_{\tau=1}^{t} \mu_\tau^t \right] \qquad \forall t \in T.$$

We can drop the minimization terms of the reformulated constraints. Hence, we obtain the final reformulation of the robust counterpart under the budgeted uncertainty set ($\mathrm{RC}_{\mathcal{U}}$) that is given by the following:

$$\min \qquad \sum_{t \in T} s_t Y_t + v_t X_t + H_t \tag{8}$$

$$s.t. :$$

$$H_t \geq h_t \left[\sum_{\tau=1}^{t} (\bar{\rho}_\tau X_\tau - d_\tau + \mu_\tau^t) + \lambda_t \Gamma_t \right] \qquad t \in T \tag{9}$$

$$H_t \geq -b_t \left[\sum_{\tau=1}^{t} (\bar{\rho}_\tau X_\tau - d_\tau - \mu_\tau^t) - \lambda_t \Gamma_t \right] \qquad t \in T \tag{10}$$

$$\lambda_t + \mu_\tau^t \geq \hat{\rho}_\tau X_\tau \qquad t \in T;\ \tau \leq t \tag{11}$$

$$X_t \leq M Y_t \qquad t \in T$$

$$X_t, H_t, \lambda_t \geq 0 \qquad t \in T$$

$$\mu_\tau^t \geq 0 \qquad t \in T;\ \tau \leq t$$

$$Y_t \in \{0, 1\} \qquad t \in T$$

3.5 Properties of an optimal robust policy

In this section, we first define the optimal policies for the inventory management problem based on the special case defined by a box uncertainty set ($\mathcal{U}_\infty = \{\mathbf{Z} \in \mathbb{R}^T : \sum_{\tau=1}^{t} |Z_\tau| \leq T \ \ t \in T\}$) and a stationary case, here when we have a constant nominal value $\bar{\rho}_t = \bar{\rho}$ and a constant maximum deviation $\hat{\rho}_t = \hat{\rho}$ of the yield. We later extend these policies for the non stationary case when the nominal value and the maximum deviation are not constant. We also give some insights about extending these policies for the LSPs, when the unit production costs are take into account.

Proposition 3.1. For the stationary case of a single-period inventory management problem under yield uncertainty with a box uncertainty set, the optimal robust policy is as follows:

$$X = \frac{d}{\bar{\rho} + \hat{\rho}\left(\frac{h-b}{h+b}\right)}$$

Proof. For this special case, the inventory $I(X, Z)$ and backorder $B(X, Z)$ costs depend on the lot size X and the disturbance Z from the mean:

$$I(X, Z) = \max\{h\,[X(\bar{\rho} + \hat{\rho}Z) - d]\ ,\ 0\}$$
$$B(X, Z) = \max\{b\,[d - X(\bar{\rho} + \hat{\rho}Z)]\ ,\ 0\}$$

Observe that the worst-case situation corresponds to $|Z| = 1$, with the worst inventory costs $I(X, Z = 1)$ and worst backorder cost $B(X, Z = -1)$. The total cost $(I(X, Z = 1) + B(X, Z = -1))$ is piecewise linear convex in X, which reaches its minimum in the period t where $I(X, Z = 1) = B(X, Z = -1)$. Therefore, we have the following:

$$h\,[X(\bar{\rho} + \hat{\rho}Z) - d] = b\,[d - X(\bar{\rho} + \hat{\rho}Z)]$$

that results in our optimal robust policy $X = \frac{d}{\bar{\rho}+\hat{\rho}\left(\frac{h-b}{h+b}\right)}$. □

Proposition 3.1 can be easily adapted to account for the unit production cost and budgeted uncertainty set. Regarding the unit production cost, the policy given by Proposition 3.1 remains valid if $v \leq (\bar{\rho} - \hat{\rho})b$, and the optimal production quantity becomes 0 if $v > (\bar{\rho} - \hat{\rho})b$. The variable costs do not have any impact on the optimum value of the policy because it is a linear function added to the piecewise linear cost function representing the inventory and backorder costs. Therefore, the the function H defining the minimum cost $H{:}h[X(\bar{\rho}+\hat{\rho}Z)-d]-b[d-X(\bar{\rho}+\hat{\rho}Z)]$ has an upward shift equivalent to the production costs. Thus, H becomes $H{:}vX+h[X(\bar{\rho}+\hat{\rho}Z)-d]-b[d-X(\bar{\rho}+\hat{\rho}Z)]$. H remains piecewise linear convex, and the minimum remains at the breakpoint. However, if the unit production cost becomes more expensive than the cost to backorder demands on the worst-case realization of the uncertain yield, it becomes more expensive to produce and keep the remaining goods in stock. As a result, it is more profitable to backorder the demands without producing goods. Note that Proposition 3.1 can easily be adapted to the budgeted uncertainty set. Because this polyhedral uncertainty set only restricts the range of the maximum deviation, the worst-case scenario changes from $|Z| = 1$ to $|Z| = \Gamma$. As a result, to consider the optimal policy for the budgeted uncertainty set in the single-period model, it is sufficient to replace $\hat{\rho}$ by $\hat{\rho}\Gamma$.

The rest of this section extends our analysis to the multiperiod inventory management problem under yield uncertainty.

Proposition 3.2. For the stationary case of the multiperiod inventory management problem under yield uncertainty with box uncertainty set, the optimal robust policy is as follows:

$$X_t = \frac{d_t}{\bar{\rho} + \hat{\rho}\left(\frac{h-b}{h+b}\right)}$$

Proof. As in the proof of the single-period problem, we define the inventory and backorder costs as follows:

$$I_t(X_t, Z_{1\ldots t}) = \max\left\{h\sum_{\tau=1}^{t}\left[X_\tau(\bar{\rho} + \hat{\rho}Z_\tau) - d_\tau\right]\ ;\ 0\right\}$$
$$B_t(X_t, Z_{1\ldots t}) = \max\left\{b\sum_{\tau=1}^{t}\left[d_\tau - X_\tau(\bar{\rho} + \hat{\rho}Z_\tau)\right]\ ;\ 0\right\}$$

where $Z_{1\ldots t}$ is the vector of Z_τ for all $\tau \in \{1 \ldots t\}$. Because we consider a constraint-wise RO, the values of Z_t are chosen independently in each period. Therefore, the worst-case situation in a period t sets Z_τ to 1 (for inventory) or -1 (for backorder) for all $\tau \in \{1 \ldots t\}$. Let us define the cumulative demand as $\bar{D}_t = \sum_{\tau=1}^{t} d_\tau$ and the cumulative production quantity up to period t as $\bar{X}_t = \sum_{\tau=1}^{t} X_\tau$. The inventory and backorder cost in period t can be written as follows:

$$I_t(\bar{X}_t, Z_{1\ldots t} = 1) = \max\left\{h\left[\bar{X}_t(\bar{\rho} + \hat{\rho}) - \bar{D}_t\right]\ ;\ 0\right\}$$
$$B_t(\bar{X}_t, Z_{1\ldots t} = -1) = \max\left\{b\left[\bar{D}_t - \bar{X}_t(\bar{\rho} - \hat{\rho})\right]\ ;\ 0\right\}$$

and the total cost $(I_t(\bar{X}_t, Z_{1\ldots t} = 1) + B_t(\bar{X}_t, Z_{1\ldots t} = -1))$ is piecewise linear convex in X, and it reaches its minimum in the period t where $I_t(\bar{X}_t, Z_{1\ldots t} = 1) = B_t(\bar{X}_t, Z_{1\ldots t} = -1)$.

The optimal robust cumulative policy for each period t is given by the following:

$$\bar{X}_t = \frac{\bar{D}_t}{\bar{\rho} + \hat{\rho}\left(\frac{h-b}{h+b}\right)} = \frac{\sum_{\tau=1}^{t} d_\tau}{\bar{\rho} + \hat{\rho}\left(\frac{h-b}{h+b}\right)}$$

Because the production in period t equals the difference between the accumulated production up to t and up to period $t-1$, the optimal robust policy for the multiperiod problem is as follows:

$$X_t = \bar{X}_t - \bar{X}_{t-1} = \frac{\bar{D}_t - \bar{D}_{t-1}}{\bar{\rho} + \hat{\rho}\left(\frac{h-b}{h+b}\right)} = \frac{\sum_{\tau=1}^{t} d_\tau - \sum_{\tau=1}^{t-1} d_\tau}{\bar{\rho} + \hat{\rho}\left(\frac{h-b}{h+b}\right)} = \frac{d_t}{\bar{\rho} + \hat{\rho}\left(\frac{h-b}{h+b}\right)}$$

□

Much like the single-period policy, Proposition 3.2 applies to the case where a constant unit production cost v is considered if $v \leq (\bar{\rho} - \hat{\rho})b$, yielding a total production cost $vX : vX \leq b(\bar{\rho} - \hat{\rho})X$. This condition indicates that it is profitable to produce because the unit cost of production is lower than the backorder cost under the worst-case scenario. Otherwise, $vX > b(\bar{\rho} - \hat{\rho})X$, so not meeting the demands would minimize expenses. It is not trivial to extend the multiperiod lot-sizing policy to the non stationary case, where the nominal value and maximum deviation of the production yield are not constant. When the production yield varies, it might be preferable to backorder (or carry inventory) in every period when the worst-case production yield rate is maximal. As a consequence, we cannot provide a closed-form solution, but the reader can rely on the MILP model and MILP solver to efficiently solve the problem.

The notable challenges in solving the multiperiod problem analytically lies in the fact that there is a direct influence of a realization of the uncertain yield on the production quantity decisions in the subsequent periods. More specifically, for a given period t, the production quantity is defined based on the impact of the uncertain yield on the decision. However, this parameter is measured only after the realization of the production yield. For this reason, it is too complex to directly incorporate variable unit production costs, non constant nominal values, and maximum deviations of the yield into the multiperiod problem. To solve this analytically, it would be necessary to prove that (i) for each period, the optimal production quantity remains valid if the unit production cost v is lower or equal to $(\bar{\rho}-\hat{\rho})b$, (ii) the inventory and backorder cost functions remain convex when the optimal production quantity for period t is derived from the optimal production values that are computed for the periods up to t .

An analytical solution for the multiperiod problem under the budgeted uncertainty set is also not trivial. The budget prevents setting all the Z_t to -1 or 1, and the worst-case scenario will set the disturbance Z_t to -1 or 1 for the largest production quantity X_t. Because Z_t depends on the production quantity and because the budget Γ can lead to a worst-case scenario where $|Z_t|$ is lower than one, it is complex to prove the convexity of the inventory and backorder cost functions under this set. As a result, it becomes difficult to derive a closed-form solution for X_t. For this case, the solution can be obtained using the robust optimization model for the LSP with an uncertain yield, as proposed in Section 3.4, which can be efficiently solved by a MILP solver.

3.6 Dynamic programming for a robust LSP with uncertain yield

This section presents a dynamic programming algorithm based on the optimal robust policy presented in Section 3.5 for solving a multiperiod LSP under yield uncertainty for a stationary case with box uncertainty set ($\Gamma = T$), constant nominal value ($\bar{\rho}_t = \bar{\rho}$), and constant maximum deviation ($\hat{\rho}_t = \hat{\rho}$) of the yield when the setup cost s_t is also considered. The proposed dynamic programming algorithm extends the method of [52] that computes a solution from a succession of regeneration intervals. Although the original dynamic programming proposed by [52] considers an uncertain demand, it does not allow for a positive backorder at the end of the production horizon, defining a regeneration point as a period where the inventory is zero or becomes negative. Our extension for the case of uncertain yield allows the backorder at the end of the production horizon, and it defines the regeneration point in terms of the inventory and backorder costs. Proposition 3.3 shows that the solution to the robust LSP under uncertain yield is also a succession of regeneration intervals, but the regeneration point is the period where the worst-case backorder cost equals the worst-case inventory cost. Before introducing Proposition 3.3, we note that the inventory and backorder cost in period τ, which are both given by $H_\tau(\bar{X}_\tau)$, depends on the cumulative production in period τ because the worst-case inventory (resp. backorder) cost corresponds to $Z = 1$ (resp. $Z = -1$):

$$H_\tau(\bar{X}_\tau) = \max \begin{Bmatrix} h\left[\bar{X}_\tau(\bar{\rho}+\hat{\rho}) - \bar{D}_t\right] \\ b\left[\bar{D}_t - \bar{X}_\tau(\bar{\rho}-\hat{\rho})\right] \end{Bmatrix}$$

Proposition 3.3. The amount $\bar{X}_{\beta_1}$ to produce in period β_1 is such that the inventory and the backorder costs are equal in some period γ_1. γ_1 is a regeneration point, and the quantity $\bar{X}_{\beta_1}$ is calculated as follows:

$$\bar{X}_{\beta_1} = \frac{\bar{D}_{\beta_1}}{\bar{\rho} + \hat{\rho}\left(\frac{h-b}{h+b}\right)} = \frac{\sum_{\tau=1}^{\beta_1} d_\tau}{\bar{\rho} + \hat{\rho}\left(\frac{h-b}{h+b}\right)}$$

Proof. Given two consecutive periods with setup β_1 and β_2 (with no production in periods $\beta_1 + 1, ..., \beta_2 - 1$), H in the interval $[\beta_1, \beta_2 - 1]$ depends only on the cumulative production $\bar{X}_{\beta_1}$ in period β_1. In addition, $\sum_{\tau=\beta_1}^{\beta_2} H_\tau(\bar{X}_{\beta_1})$ is a piecewise linear and convex function (see Figure 1), such that the minimum is at a breakpoint. Each breakpoint corresponds to the case were the backorder cost equals the inventory cost in a period. □

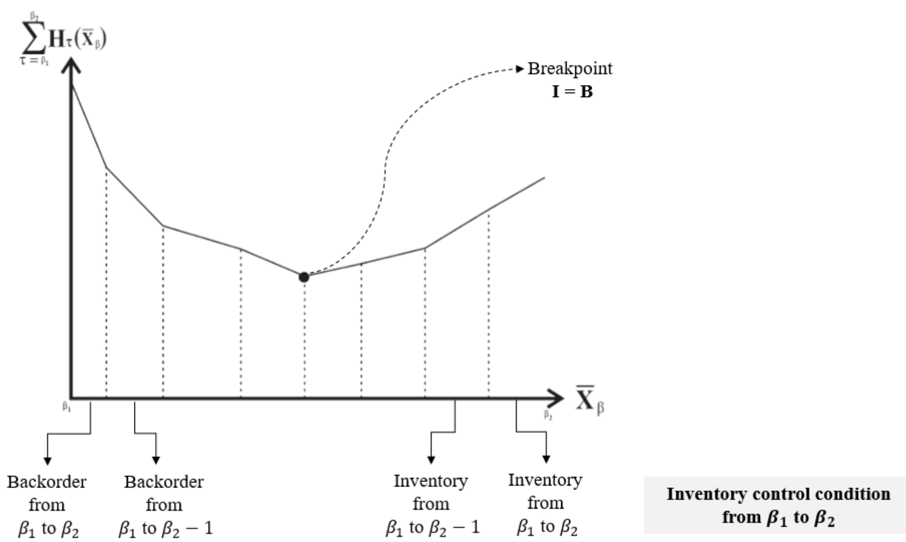

Figure 1: Piecewise inventory and backorder cost functions

Let α be the last regeneration point before starting a production lot in the period β, such that γ is the next regeneration point after α. Figure 2 illustrates the regeneration point and periods covered by the production in period β. Here, I (resp. B) indicates that the worst-case cost in the period and corresponds to the inventory (resp. backorder) cost. The worst-case cost is the backorder cost from period α to $\beta - 1$, and it is the inventory cost from $\beta + 1$ to $\gamma - 1$.

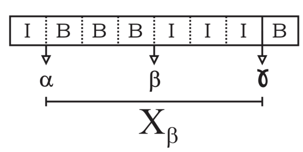

Figure 2: Representation of the regeneration point concept

The dynamic programming algorithm selects the optimal set of regeneration points. On contrary to the deterministic case, it is possible to obtain a plan whose level of backorder at the end of the production horizon is non zero for the dynamic program for the LSP under yield uncertainty. For this reason, we can define three optimal cost functions. Let $F(t)$ be the optimal cost at the period t, $G(\alpha, \gamma)$, which is the optimal cost from the beginning of the period α to the end of the period γ when both the inventory and backorder levels are considered, and $K(\alpha, T)$ is the optimal cost from the beginning of the period α to the end of the planning horizon, when only the backorder levels are considered. Here, $K(\alpha, T)$ represents the possibility of backordered items at the end of the production horizon. We can recursively compute $F(t)$, $G(\alpha,\gamma)$, and $K(\alpha,T)$ as follows:

$$F(t) = \min_{\alpha \le t} \{F(\alpha-1) + G(\alpha,t)\} \quad \forall\, t \le T-1$$

$$F(T) = \min\left\{ \min_{\alpha \le T} \{F(\alpha-1) + G(\alpha,T)\} \;;\; \min_{\alpha \le T} \{F(\alpha-1) + K(\alpha,T)\} \right\}$$

$$G(\alpha,\gamma) = \min_{\alpha \le \beta \le \gamma} \left\{ s_\beta + \sum_{\tau=\alpha}^{\beta-1} H_\tau(\bar{X}_{\alpha-1}) + \sum_{\tau=\beta}^{\gamma} H_\tau(\bar{X}_\beta) \right\}$$

$$K(\alpha,T) = \sum_{\tau=\alpha}^{T} H_\tau(\bar{X}_{\alpha-1})$$

where $F(0) = 0$.

Knowing that the complexity for the minimum and sum functions is given by $O(n)$, we can demonstrate that the complexity for our functions $G(\alpha,\gamma)$ and $K(\alpha,\gamma)$ are $O(T^2)$ and $O(T)$, respectively. Therefore, we have the following:

$$O(F(T)) = O(2) \cdot \{[O(T) \cdot (O(T^2) + O(T^2))] + [O(T) \cdot (O(T^2) + O(T))]\}$$

Hence, the complexity of our robust dynamic program is $O(T^3)$. The problem becomes NP-Hard when extended to the capacitated context, and this can be verified with a reduction from the deterministic single-period capacitated LSP [53, 54]. Much like for the policy for the multiperiod inventory management problem, the consideration of a budgeted uncertainty set is too complex. In fact, it is not trivial to demonstrate the convexity of the value functions under the budgeted uncertainty set, and without such proof, we are not able to obtain a closed form to compute a production plan. In this case, we can instead adopt the MILP robust model to determine a production plan.

4 Results and discussions

This section presents the experimental study, and its objective is threefold: (1) to demonstrate the robustness of the optimization methods and approaches to cope with a nondeterministic LSP; (2) to present an in-depth investigation on the robust LSP with uncertain yield, its performance, the quality of the solutions, and the computational efficiency; and (3) to evaluate and compare the performance of the different optimization approaches in terms of the average and worst-case quality of the solution.

The experiments consider the following solution approaches:

1. DET, the nominal model with $\rho_t = \bar{\rho}_t$
2. $SP_{uniform}$, the stochastic program with 500 scenarios, where the yield realizations are randomly drawn from a uniform distribution with support $[\bar{\rho}_t - \hat{\rho}_t; \bar{\rho}_t + \hat{\rho}_t]$ for each period t
3. RO_Γ, the budgeted-uncertainty set robust optimization with budget Γ
4. RO_{DP}, the dynamic program based on the robust budgeted uncertainty set with budget $\Gamma = T$ for the special case presented in Section 3.6
5. RO^*_Γ, the robust model for the special case of the dynamic program presented in Section 3.6

Note that the nominal problem and stochastic program are natural benchmarks to compare solution approaches that cope with uncertainties. This section is organized as follows: Section 4.1 presents the instances generation method. Section 4.2 introduces the simulation framework used to compare the methods. Finally, Section 4.3 presents an analysis of the developed models. We discuss the performance and quality of the plans resulting from different optimization approaches, and we highlight the advantages of using robust optimization to hedge against uncertainties in a highly uncertain context.

4.1 Instance generation

We generate the instances muck like [30] and [55]. To adapt these instances for a LSP under yield uncertainty, the production costs, the inventory costs, the demands, the nominal values of the production yield rate, and the maximum deviations of the production yield are randomly generated using a normal distribution within the following intervals: $v_t \in U(10,20)$, $h_t \in U(1,10)$, $d_t \in U(140,480)$, $\bar{\rho}_t \in U(0.5,0.7)$, and $\hat{\rho}_t \in U(0.1,0.3)$, respectively. The setup costs are computed with the time between order formula: $s_t = \frac{\bar{D}_t \cdot TBO^2 \cdot h_t}{2}$, where $\bar{D}_t$ represents the average demand up to period t. For the capacitated case, we derive the capacity from the value M of the uncapacitated case (i. e., $M = \frac{\sum_{t \in T} d_t}{\min_{t \in T} \bar{\rho}_t}$) and multiply it with a factor cp. Therefore, the capacity can be computed as $C = cp \frac{\sum_{t \in T} d_t}{\min_{t \in T} (\bar{\rho}_t - \hat{\rho}_t)}$. We consider instances with 4, 12, and 24 periods, a time between orders of 1, 2, or 4, and a backorder cost that equals 2, 5, or 10 times the holding cost for each period t. In addition, we consider a capacity factor of 25%, 50%, and 75% for the capacitated model. We generate the instances with a full factorial design for these four parameters, which leads to 108 instances. We set the inventory and backorder levels at the beginning of the horizon to zero. Because the optimal policies are valid only for the uncapacitated models, we generate 18 instances resulting from the factorial design of the following parameters: 4, 12, and 24 periods, time between orders of 1 or 2, and backorder costs equaling $2h_t$, $5h_t$, or $10h_t$.

4.2 Simulation

We analyze the quality of the production plans through a simulation with $|\Omega| = 5000$ scenarios generated with Monte Carlo sampling, where each scenario represents a possible realization of the production yields over the horizon. We simulate the yield with a uniform distribution with support $[\bar{\rho}_t - \hat{\rho}_t; \bar{\rho}_t + \hat{\rho}_t]$ in period t. We note $EVPI$, the expected value of perfect information, the average cost of the perfect information solutions, where $EVPI_\omega$ is the cost of the solution computed with the deterministic model for scenario ω. To evaluate each optimization method, we fix X_t and Y_t obtained from the optimization step in the deterministic model for each scenario ω.

4.3 Experimental results

This section presents an average and worst-case cost analysis for the uncapacitated and capacitated problems. We compare the methods based on the objective value provided by each optimization approach (e.g., the objective function given in (4) for RO_Γ), the average computational time (in seconds), the expected value (Exp. Cost) of each solution approaches evaluated in the simulation, along with the worst-case cost in the simulation, and the 95^{th} and 99^{th} percentile cost (p.c.). We also report the relative difference between the expected value of perfect information $EVPI$ and the simulated expected value of each method given by (12). In addition, we define in (13) the relative difference between the objective value of a solution approach and its simulated expected cost. Finally, we report the coefficient of variation CV, an index that indicates a high variability of the costs in the simulation. Thus, CV is the ratio of the standard deviation to the mean, such that the higher CV is, the more dispersed the values from the mean are.

$$GAP_{EVPI} = 100 \times \frac{\textbf{Exp. Cost} - \textbf{EVPI}}{\textbf{EVPI}} \tag{12}$$

$$GAP_{OPT} = 100 \times \frac{\textbf{Exp. Cost} - \textbf{Obj. Value}}{\textbf{Obj. Value}} \tag{13}$$

4.3.1 Solutions of the inventory management problem for the special case defined in Section 3.5

First, we analyze the performance of the dynamic program to solve the multi period LSP under yield uncertainty for the stationary case with the box uncertainty set ($\Gamma = T$), constant nominal value and maximum deviation of the production yield, and setup cost. We compare the robust dynamic program RO_{DP} to the robust model $RO^*_{\Gamma=T}$ applied to this special case.

As reported in Table 1, the robust dynamic program is much less demanding than the robust model in terms of computational efforts. For the special case, RO_{DP} $RO^*_{\Gamma=T}$ provides the same production plan. The dynamic program takes approximately 0.04 seconds to propose a solution, while the robust model needs approximately 0.26 seconds to compute the same solution. The dynamic program is easily implementable, does not require an MILP solver, and has a computation that is much easier to understand than the mathematical formulation of the robust model. However, the dynamic program holds only for the special case. On the other hand, the robust model is efficient when dealing with more general LSPs because it requires no strong assumptions about the characteristics of the decision context. In addition, the robust model can be extended to include different practical constraints.

Table 1: Performance of the dynamic programming approach and reformulated MILP

Model	Exp. Cost	95^{th} p.c.	99^{th} p.c.	Worst Cost	Obj. Value	Comp. Time (s)	C_v
RO_{DP}	133,316	155,659	164,295	180,470	202,623	0.04	0.06
$RO^*_{\Gamma=T}$	133,316	155,659	164,295	180,470	202,623	0.26	0.06

4.3.2 Price of robustness

To analyze the impact of the budget Γ, we consider different budgets of uncertainties to represent the decision-maker's risk aversion. For this, we take from low aversion ($\Gamma = 1$) to extreme aversion $\Gamma = T$, going through a progression with the size of the production planning. In fact, because the budget represents the maximum number of uncertain parameters that can take the worst-case value [49], Γ indicates the number of periods where the production yield can take its worst-case realization. Thus, it is convenient to express Γ as a proportion of T. Table 2 indicates the impact of the budget of uncertainty on the costs. While the Objective Value column gives the average of the objective function values computed in the optimization step, the remaining columns report the average features obtained in the simulation step.

Table 2: Impact of the budget of uncertainty on the robust lot-sizing decision

	Uncapacitated						Capacitated					
Γ	Obj Value	Exp. Cost	95^{th} p.c.	99^{th} p.c.	Worst Cost	CV	Obj Value	Exp. Cost	95^{th} p.c.	99^{th} p.c.	Worst Cost	CV
1	202,644	190,277	232,106	270,473	345,261	12%	345,651	330,301	382,158	414,208	474,505	13%
0.1T	202,644	190,277	232,106	270,473	345,261	12%	345,651	330,301	382,158	414,208	474,505	13%
0.2T	243,053	204,576	228,261	244,842	294,790	17%	385,138	335,729	372,465	392,821	441,631	17%
0.3T	271,007	214,963	236,399	248,710	278,328	19%	414,691	342,376	376,022	392,452	426,458	20%
0.4T	302,845	227,093	250,816	260,701	280,634	20%	451,912	350,948	384,520	398,820	426,080	21%
0.5T	314,929	227,650	254,427	264,686	284,989	18%	472,655	355,476	389,544	403,796	430,113	21%
0.6T	327,803	230,016	259,966	270,597	289,004	16%	498,531	361,949	396,974	410,826	435,796	20%
0.7T	334,611	232,977	263,434	274,430	293,206	15%	511,743	363,433	398,491	412,496	438,421	20%
0.8T	340,519	233,412	266,356	278,026	298,857	14%	526,262	364,423	399,938	414,275	440,672	19%
0.9T	342,273	232,509	265,815	277,846	299,082	14%	531,739	364,424	400,286	414,681	441,325	19%
T	339,353	229,149	262,188	274,229	295,838	13%	533,134	362,977	398,872	413,335	440,240	19%

As reported in Table 2, for both the uncapacitated and capacitated model, the best solutions in terms of objective value and expected cost are obtained for a very tiny budget (Γ equals 1, $0.1T$ and $0.2T$), so the decision-maker should be willing to accept a high degree of risks resulting from uncertainties. Considering the 95^{th} percentile costs, the lowest costs for the uncapacitated (resp. capacitated) model are obtained for Γ between 1 and $0.3T$ (resp. $0.2T$ and $0.3T$), while the lowest 95^{th} percentile costs are given by a budget equals $0.2T$ and $0.3T$ (resp. $0.2T$, $0.3T$, and $0.4T$). In addition, the lowest worst case costs are obtained for Γ between $0.2T$ and $0.7T$ (resp. $0.3T$ and $0.7T$), and the lowest coefficient of variation CV can be achieved for a budget that equals 1, $0.1T$ or between $0.8T$ and $0T$ (resp. 1 and $0.1T$). However, for Γ in the range between 1 and $0.1T$, the robust objective value fails to bound the 95^{th} percentile cost (and, consequently, the worst-case cost). Therefore, with low Γ values, the decision-maker has no guarantee that the plan will be sufficiently immunized from uncertainties. On the other hand, although the solutions obtained for Γ that are greater than $0.4T$ achieve a robustness level that covers the worst-case scenarios, they are too conservative. In fact, for Γ between $0.5T$ and T, the objective value is much higher than the worst-case cost. Thus, even if for the largest value of Γ the solutions are robust, they do not result in the best option in terms of the expected costs and stability. Therefore, the budget of uncertainty in the interval from $0.1T$ to $0.4T$ offers better production plans because the decisions are sufficiently stable because of a low CV, the expected cost is relatively low, and the robust objective value covers at least the 95^{th} percentile costs (and even the 99^{th} percentile costs for when Γ equals $0.3T$ or $0.4T$.

4.3.3 Comparison with the stochastic programming plan

We now compare the production plan resulting from the robust model, stochastic program, and deterministic model. We consider RO with a budget that equals $0.2T$, $0.3T$, and $0.4T$ because we previously have shown that these values result in better cost and robustness trade-offs. In addition, we consider the extreme cases of the robust approach, that is, we either neglect the risk ($\Gamma = 0$ also represented by the nominal problem) or consider the most conservative solution ($\Gamma = T$).

Table 3 shows the simulation results for the uncapacitated LSP. Here, the nominal solution is the least robust because its average worst-case cost is the largest among all the tested methods. The computation time required by the nominal problem is low, but it results in a solution that is not sufficiently robust or optimal in the nondeterministic context.

Table 3: Performance of the uncapacitated models in terms of the average cost and worst-case simulated costs

Model	Exp. Cost	95^{th} p.c.	99^{th} p.c.	Worst Cost	Comp. Time	CV	$\mathbf{GAP_{EVPI}}$	$\mathbf{GAP_{OPT}}$
$EVPI$	159,080	175,857	183,418	197,468	-	18%	-	-
DET	182,234	263,257	308,137	388,106	0.08	7%	15%	10%
SP	180,065	228,042	263,070	338,144	27.53	9%	13%	-3%
$RO_{\Gamma=0.2T}$	190,277	232,106	270,473	345,261	0.33	12%	20%	-6%
$RO_{\Gamma=0.3T}$	204,576	228,261	244,842	294,790	1.37	17%	29%	-16%
$RO_{\Gamma=0.4T}$	214,963	236,399	248,710	278,328	2.01	19%	35%	-21%
$RO_{\Gamma=T}$	232,509	265,815	277,846	299,082	0.04	14%	46%	-32%

Because the nominal model disregards the impact of uncertainties on the decision, we focus on the robust and stochastic optimization methods. Even if the lowest expected costs are obtained by SP and $RO_{\Gamma=0.2T}$ and the lowest 95^{th} percentile costs are given by SP and $RO_{\Gamma=0.3T}$, the robust models for Γ that equals $0.3T$ and $0.4T$ have the lowest costs for the 99^{th} percentile costs and the worst-case costs. Although the stochastic program provides lower expected costs (180,065) than the robust approach (210,581, on average), the robust model leads to a lower worst-case cost (about 304,365 on average over the different budgets) than the worst cost obtained with the stochastic plan (338,144). Nevertheless, we stress that the SP and RO methodologies have fundamentally different objectives. Although the stochastic program seeks the minimum expected costs, the robust optimization method optimizes over the minimum worst-case costs. This is why the difference between the optimization gap of both approaches is so important. In the same vein, the relative difference between $EVPI$ and RO_{Γ} is greater than the $\mathbf{GAP_{EVPI}}$ between $EVPI$ and SP because of the robust strategy to propose a production plan that remains feasible even for the worst realization of the uncertain yield. This strategy leads to more conservative planning than the production plan proposed by the stochastic program, for which the strategy is defined regarding the probability of the realization of the uncertainty. However, SP is known to be prone to changes in the underlying uncertainty (e.g., if the distribution changes), while the robust approach remains stable and robust for different and unrelated uncertainty realizations. This is confirmed with the 99^{th} percentile and worst-case average costs, for which robust models are much less impacted by the uncertain parameter, leading to lower costs.

Table 4 presents the results for the capacitated version of the LSP. Much like the uncapacitated model, the lowest expected cost is given by SP. However, the lowest 95^{th} percentile, 99^{th} percentile and worst-case costs are given by the robust models (especially for when Γ equals $0.3T$ and $0.4T$). In addition, the coefficient of variation of the robust models decreases in comparison with the uncapacitated version of the problem, such that SP and $RO_{\Gamma=0.2T}$ achieve the same CV. In addition, the relative difference between the optimization methods and $EVPI$ becomes lower, and the optimality gap for the robust models also decreases. The results show that the robust approach is efficient when mitigating uncertainties because it offers a good and robust decision plan. The expected costs computed for each method are very similar. From a budget greater than 0.3T, the 99^{th} percentile cost is usually covered by the optimal value. In addition, from a budget greater than or that equals 0.4T, even the worst-case cost is covered by the objective value. As a result, we note that the SP model is less efficient when uncertainty information is relatively limited or if we want to limit a downside risk because of the realization of uncertainties.

Table 4: Performance of the capacitated models in terms of the average cost and worst-case simulated costs

Model	Exp. Cost	$95^{\mathbf{th}}$ p.c.	$99^{\mathbf{th}}$ p.c.	Worst Cost	Comp. Time	CV	$\mathbf{GAP_{EVPI}}$	$\mathbf{GAP_{OPT}}$
$EVPI$	317,152	352,486	366,866	393,076	-	17%	-	-
DET	327,925	398,981	434,486	496,290	0.04	11%	3%	2%
SP	327,324	375,637	405,292	464,983	6.42	13%	3%	-1%
$RO_{\Gamma=0.2T}$	330,301	382,158	414,208	474,505	0.03	13%	4%	-4%
$RO_{\Gamma=0.3T}$	335,729	372,465	392,821	441,631	0.05	17%	6%	-13%
$RO_{\Gamma=0.4T}$	342,376	376,022	392,452	426,458	0.07	20%	8%	-17%
$RO_{\Gamma=T}$	364,424	400,286	414,681	441,325	0.01	19%	15%	-31%

In Table 3 (resp. Table 4), the Comp. Time column reports the average computation time in seconds of each method for the uncapacitated (resp. capacitated) instances. For the uncapacitated models, the nominal problem is the least demanding to solve (approximately 0.08 seconds on average), the robust model requires little computational effort (0.94 seconds on average), and the SP model is more difficult to solve (approximately 27.53 seconds on average). For the capacitated models in Table 4, we see similar patterns. The computational efforts for the nominal and robust models are low, with an average computation time of 0.04 seconds for both of them. However, the nominal problem ignores the uncertainty, while the robust plan proposes a decision that mitigates these uncertainties. The stochastic program is more demanding in terms of computational efforts (approximately 6.42 seconds on average). Hence, a robust approach would be preferable because it proposes a good production plan that is robust and quick to calculate. The literature usually indicates that deterministic capacitated problems are more difficult to solve than their uncapacitated versions [56].[1] However, our results indicate that all approaches solve the capacitated model faster. This behavior was also observed by [58] for an LSP with demand uncertainty, where the authors conclude that a robust capacitated LSP is easier to solve than the uncapacitated version. The authors report that because the capacitated model has a bound M on the lot size lower than the natural bound, the linear relaxation of the capacitated version would be less fractional and would lead to better lower bounds.

To analyze the cost components incurred in the simulated production system, Table 5 reports the average number of setups $\|\mathbf{Y}\|$, production quantities $\|\mathbf{X}\|$, inventory $\|\mathbf{I}\|$, and backorders $\|\mathbf{B}\|$ accumulated over the entire production horizon for the uncapacitated and capacitated models. In addition, Table 5 gives the proportion of the average expected costs imputable to the setup, production, inventory, and backorder costs. The robust approach usually provides production plans with less backorder at the expense of more often producing a greater amount of goods. For instance, in the uncapacitated problem, RO produces an amount of goods relatively close to that proposed by the stochastic production plan (5,348 units on average when considering all Γ values), but the robust model leads to more frequent production (8 versus 5) and low backorder (1,453 versus 1,826). Because the robust model focuses on the worst-case scenario, it produces enough to face low-yield values. Therefore, when compared with the stochastic plan, the setup and production costs incurred from the robust plan tend to exceed the costs from the stochastic plan, while the inventory and backorder costs

[1] [56] mention that the complexity for an uncapacitated deterministic LSP as proved by [57] is $O(T \log T)$, while the complexity for the capacitated problem is $O(T^4)$ for the general case considered in [53].

are lower. Therefore, the robust plan offers more flexibility for the decision-maker to take advantage of the available resources while also reducing the impact of the uncertain events on the production plan.

Table 5: Characteristics of the solutions for the uncapacitated and capacitated models in terms of cost distribution

Model	Uncapacitated								Capacitated							
	$\Vert\mathbf{Y}\Vert$	$\Vert\mathbf{X}\Vert$	$\Vert\mathbf{I}\Vert$	$\Vert\mathbf{B}\Vert$	**Y cost**	**X cost**	**I cost**	**B cost**	$\Vert\mathbf{Y}\Vert$	$\Vert\mathbf{X}\Vert$	$\Vert\mathbf{I}\Vert$	$\Vert\mathbf{B}\Vert$	**Y cost**	**X cost**	**I cost**	**B cost**
$EVPI$	4	4,475	3,466	14	8%	66%	9%	16%	9	3,996	1,080	6,023	13%	46%	3%	38%
DET	5	4,955	3,379	1,989	7%	59%	8%	26%	9	4,143	1,172	6,397	12%	42%	2%	43%
SP	5	4,859	4,166	1,826	6%	59%	10%	25%	9	4,085	1,758	6,341	12%	42%	4%	43%
$RO_{\Gamma=0.2T}$	7	5,076	3,470	1,602	11%	60%	8%	21%	9	4,192	1,489	6,289	13%	42%	3%	42%
$RO_{\Gamma=0.3T}$	9	5,192	4,539	1,373	13%	60%	9%	18%	10	4,278	2,382	6,071	13%	42%	5%	40%
$RO_{\Gamma=0.4T}$	9	5,318	5,265	1,315	13%	59%	11%	17%	10	4,361	2,953	5,993	13%	42%	6%	40%
$RO_{\Gamma=T}$	6	5,804	9,765	1,522	6%	57%	18%	20%	9	4,463	4,994	6,556	10%	37%	8%	45%

Table 5 also shows the solutions characteristics for the capacitated model. As expected, when the availability of the resources is more restricted, backorder becomes more frequent. Thus, the robust plan manages to control (and even reduce) the backorder cost by increasing the lot size and the frequency of production setup. As a result, the robust model favors a large production level to meet demands, while the stochastic program takes the risk of backordering goods.

To conclude, the robust model provides effective support for decision-makers. Contrary to the nominal and stochastic models, the robust models provide an objective value that is larger than the expected simulated costs, and this can reassure the decision-maker. In addition, unlike other approaches, the robust plan covers even the most pessimistic scenario. When we investigate the stability and robustness of the proposed plans, the robust approaches provide the production plan that better copes with uncertainties because it tends to offer a greater cost savings with a relatively low coefficient of variation. In addition, although the robust model relies on stock to satisfy demands, the stochastic model adopts a strategy that places backorders more often to reduce the inventory and production costs. Therefore, the robust models better mitigate the impact of the realization of unknown and pessimistic scenarios on the decision made.

5 Conclusion

In the current paper, we have introduced the robust formulation for lot-sizing under yield uncertainty. We show that the stationary case of the multiperiod problem, where the average and standard deviation of the production yield rate are constant over the planning horizon, can be solved in polynomial time with a dynamic programming approach for the case of a box uncertainty set. The present paper also provides insights into robust production plans. Our results show that with a proper budget of uncertainty, the robust model mitigates uncertainties with a balance between production quantities, setup costs, and inventory management costs. In addition, the robust optimization method requires less computational effort than stochastic programming. Another major advantage of robust optimization over stochastic programming is that it requires little information about the uncertainty factors and no strong assumption on the uncertain parameter characteristic. Further investigation is still needed to propose an adaptive framework to cope with uncertainties within a static-dynamic, and even a dynamic, decision framework. The present work could also be extended to deal with multiechelon systems.

Appendices

A The two-stage stochastic programming LSP with yield uncertainty model

SP handles uncertainty through a mathematical program whose objective is to minimize the expected cost [29]. The uncertain parameter is described by a probability distribution and some statistical indicators (e.g.: mean and standard deviation) that are usually gathered by processing and analysis of data from historic data and other available data about the decision system. Therefore, the stochastic programs is a natural benchmark to compare the production plans proposed by other methodologies and verify their performance and quality.

To evaluate the performance of our robust model, we propose a scenario-based stochastic programming to represent the lot-sizing problem under yield uncertainty, based on the case study presented by [59]. We consider a set Ω of possible yield scenarios, where each scenario ω has a p_ω probability of realization, and ρ_t^ω is the realization of the uncertain yield for the period t of in scenario ω. The two-stage stochastic programming for the LSP with uncertain yield is given as follows:

$$\min \quad \sum_{\omega \in \Omega} p_\omega \sum_{t \in T} s_t Y_t + v_t X_t + h_t I_t^\omega + b_t B_t^\omega \tag{14}$$

$$\begin{aligned}
&s.t. \\
&I_t^\omega - B_t t^\omega = I_{t-1}^\omega - B_t t - 1^\omega + \rho_t^\omega X_t - d_t && t \in T \ \omega \in \Omega \qquad (15)\\
&X_t \leq M_t \cdot Y_t && t \in T\\
&X_t \geq \ 0 && t \in T\\
&I_t^\omega, B_t^\omega \geq \ 0 && t \in T; \ \omega \in \Omega\\
&Y_t \in 0, 1 && t \in T
\end{aligned}$$

Although SP is largely applied within the optimization under uncertainties, this approach often suffers from scalability issues, being computationally prohibitive, and requiring advanced techniques to generate possible scenarios. For this, as many scenarios as possible are generated in order to reflect the uncertainty distribution, even though the amount of scenarios may be limited to restrict the computational efforts.

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