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Quasi-maximum likelihood for estimating structural models

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Abstract : The structural model of Merton (1974) gives rise to multiple applications and extensions in corporate credit-risk analysis. The estimation of this framework poses a major challenge as its underlying state variable (the firm's asset value) is not directly observable. Since Duan (1994), maximum likelihood has become the benchmark to estimate structural models where corporate securities are valued in closed form. We propose a quasi-maximum likelihood (QML) approach that remains appropriate even when the explicit approach is unachievable. QML is highly flexible and effective. To assess our construction, we conduct an empirical investigation, highlight the credit-spread puzzle, and discuss a remedy via bankruptcy costs.

Keywords: Structural model, corporate securities, estimation, maximum likelihood, quasi-maximum likelihood

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1 Introduction

The aim of this paper is to estimate complex structural models used for corporate credit-risk analysis. Duan (1994) developed a maximum likelihood (ML) approach to estimate Merton's (1974) model. Since then, the literature has reported several adaptations of ML used when corporate securities can be valued in closed form. We propose a quasi-maximum likelihood (QML) approach that remains appropriate even when the explicit approach is unachievable. QML alternates between stochastic dynamic programming (SDP) and maximum likelihood (ML) to jointly solve and estimate structural settings.

The main difficulty in estimating structural models lies in the fact that their common state process (the firm's asset value) is not directly observable. The literature prior to Duan (1994 and 2000) reports several ad-hoc estimation procedures. The proxy approach combines market and accounting values of corporate securities as well as the historical and the implicit approaches to infer the model's unknown parameters (Brockman and Turtle 2003, Leland 2004, Bharath and Shumway 2008, Huang and Huang 2012, and Afik et al. 2016). The proxy approach suffers from a major drawback in that accounting values of corporate securities are not enough sensitive to changes in the firm's credit quality. The restricted volatility approach builds on equations that associate the non-observable firm's asset value and its volatility with the observable firm's equity value (market capitalization) and its volatility (Jones et al. 1984, Ronn and Verma 1986, Ogden 1987, Delianedis and Geske 2001 and 2003, Eom et al. 2004, Vassalou and Xing 2004, Hull et al. 2005, Chen et al. 2006, Charitou et al. 2013). The restricted volatility approach presents a shortcoming as it is specific to the lognormal assumption.

The ML approach of Duan (1994) has given rise to several empirical investigations (Wang and Suo 2006, Li and Wong 2008, and Lee et al. 2015) and analytical modifications under 1– Merton's (1974) model (Duan et al. 2004 and 2005, Duan and Fulop 2009, and Jovan and Ahčan 2017), 2– Leland's (1994) model (Ericsson and Reneby 2004 and 2005 and Forte and Lovreta 2012), 3– Brockman and Turtle's (2003) model (Wong and Choi 2009 and Dionne and Laajimi 2012), and 3– Black and Cox' (1976) model (Wong and Li 2006 and Amaya et al. 2019). When ML is unachievable, less demanding statistical principles are used, that is, 1– simulated ML (Bruche 2005), 2– GMM (Li et al. 2004, Hsu et al. 2010, and Huang et al. 2020), and 3– MCMC (Korteweg and Polson 2009 and Huang and Yu 2010).

We propose a quasi-maximum likelihood (QML) approach that accommodates alternative state processes and balance-sheet components. QML works as follows:

1. Derive the quasi-likelihood function;
2. Set a value for the vector of the model's unknown parameters;
3. Use a numerical procedure and solve the model (stochastic dynamic programming is used herein);
4. Extract the pseudo-time series of the firm's asset value in accordance with its associated time series of the firm's equity value;
5. Compute the value of the quasi-likelihood function;
6. Go to step 2 and repeat until the quasi-likelihoood function reaches a maximum (MADS algorithm is used herein).

QML is highly flexible and effective. To assess our construction, we use Monte Carlo simulation and show that QML and ML are consistent when the likelihood function is known in closed form. Then, we consider a Geske-like (1977) model, where ML is unachievable, and conduct an empirical investigation of a speculative-grade corporate debt, while its actual payment schedule is fully respected. We evaluate the firm's corporate securities, compute its yield spreads, and report its term structure of default probabilities. The findings show that the debt's yield spreads remain under-valued, as documented in the literature on the structural model and the credit-spread puzzle. Finally, we consider bankruptcy costs as a remedy.

The rest of this paper is organized as follows. Section 2 presents the QML approach and Section 3 reports a case study. Section 4 concludes.

2 The QML approach

2.1 The model

Let θ be the vector of estimable parameters that characterize the Markov state process A (the firm's asset value), and assume that the conditional density function of A_u given $A_t = a_t$, for $u \geq t$, indicated by $f_{A_u,\theta}(. \mid a_t)$, is a known explicit function under the physical probability measure $\mathbb{P}$. Since the firm's equity is interpreted as a financial derivative on A, its value function can be written as

$$\mathcal{E}_t(a), \quad \text{for } t \in \mathcal{P} \text{ and } a > 0,$$

where $\mathcal{P} = \left\{0, \ldots, T, \ldots T^D\right\}$ is the set of evaluation dates and $a = A_t$ the level of the state variable at time t. Dates 0 and T are the first and last observation dates of the firm's equity value (market capitalization), while date T^D is the maturity of the firm's debt portfolio. Date T also represents the present date. This scheme refutes new debt issuance over $\left[0, T^D\right]$.

We propose a structural model that accommodates arbitrary interest and capital payment schedules, multiple seniority classes, and several intangible corporate securities. For all $t \in \mathcal{P}$ and $a = A_t$, the firm's balance-sheet equality verifies

$$a + \mathrm{TB}_t(a) - \mathrm{BC}_t(a) = D_t^s(a) + D_t^j(a) + \mathcal{E}_t(a), \tag{1}$$

where TB represents tax benefits, BC bankruptcy costs, D^s the senior debt, D^j the junior debt, $D^s + D^j = D$ the debt portfolio, and $\mathcal{E}$ the firm's equity. Tax benefits, bankruptcy costs, and the total value of the firm, that is, the left-hand side of eq. (1), are defined consistently with Leland (1994). We use stochastic dynamic programming (SDP), solve the model under the risk-neutral probability measure, and provide (convergent) piecewise linear interpolations for the value functions of these corporate securities, indicated by

$$\widehat{\mathrm{TB}}, \ \widehat{\mathrm{BC}}, \ \widehat{D}, \text{ and } \widehat{\mathcal{E}}.$$

For a fixed θ, the SDP value function of the firm's equity at time $t \in \mathcal{P}$ and $a = A_t$ can be written as

$$\widehat{\mathcal{E}}_t(a) = \sum_{i=0}^{p} (\alpha_i + \beta_i a) \, \mathbb{I}(x_i \leq a < x_{i+1}), \tag{2}$$

where the x_i's are (known) evaluation nodes, the α_i's and β_i's are the (known) local coefficients of the linear interpolation $\widehat{\mathcal{E}}_t$, and $\mathbb{I}$ is the indicator function. For ease of notation, we omit to index the α_i's, β_i's, and the x_i's by θ and $t \in \mathcal{P}$. SDP is highly flexible in that it accommodates alternative balance-sheet components and state-space dynamics (Ayadi et al. 2016 and Ben-Ameur et al. 2016 and 2020).

The default event at time $t \in \mathcal{P}$ can be expressed as

$$e = \mathcal{E}_t(a) = 0 \quad \text{if, and only if,} \quad a = A_t \leq b_t, \tag{3}$$

where $b_t \in \mathbb{R}_+$ is the endogenous default barrier at time $t \in \mathcal{P}$. The default barriers are null at non-payment dates. For $e > 0$ and $a = \mathcal{E}_t(e)^{-1} > b_t$, the one-to-one transformation $\mathcal{E}_t$ gives

$$e = \mathcal{E}_t(a) \quad \text{if, and only if,} \quad a = \mathcal{E}_t(e)^{-1}. \tag{4}$$

The distribution of $\mathcal{E}_t$ is continuous everywhere except on zero, where the mass function indicates the probability of default.

2.2 The likelihood function

The likelihood function associated to the observed time series $(e_0, \ldots, e_T)$ of the Markov process $\mathcal{E}$ is

$$\begin{aligned}
&\mathcal{L}\left(\theta \mid e_0, \ldots, e_T\right) \\
&= f_{\mathcal{E}_0,\theta}\left(e_0\right) \times \prod\nolimits_{t=1}^{T} f_{\mathcal{E}_t,\theta}\left(e_t \mid e_{t-1}\right) \\
&= \prod\nolimits_{t=1}^{T} f_{\mathcal{E}_t,\theta}\left(e_t \mid e_{t-1}\right),
\end{aligned}$$

where $f_{\mathcal{E}_t,\theta}\left(. \mid e_{t-1}\right)$ is the (unknown) conditional density function of $\mathcal{E}_t$ given $\mathcal{E}_{t-1} = e_{t-1}$ under the physical probability measure, with the convention that $f_{\mathcal{E}_0,\theta}\left(e_0\right) = 1$. Since only surviving companies are observed in stock markets, the terms $f_{\mathcal{E}_t,\theta}\left(0 \mid e_{t-1}\right)$, for $t \in \mathcal{P}$, never contribute to the likelihood function.

For $e > 0$, the conditional cumulative density function of $\mathcal{E}_t$ given $\mathcal{E}_{t-1} = e_{t-1}$ verifies

$$\begin{aligned}
&F_{\mathcal{E}_{t,\theta}}\left(e \mid e_{t-1}\right) - F_{\mathcal{E}_{t,\theta}}\left(0 \mid e_{t-1}\right) \\
&= F_{\mathcal{E}_{t,\theta}}\left(e \mid e_{t-1}\right) - F_{\mathcal{E}_{t,\theta}}\left(0 \mid e_{t-1}\right) \\
&= \mathbb{P}\left(0 < \mathcal{E}_t \leq e \mid e_{t-1}\right) \\
&= \mathbb{P}\left(b_t < A_t \leq \mathcal{E}_t\left(e\right)^{-1} \mid a_{t-1}\right) \\
&= F_{A_{t,\theta}}\left(\mathcal{E}_t\left(e\right)^{-1} \mid e_{t-1}\right) - F_{\mathcal{E}_{t,\theta}}\left(0 \mid e_{t-1}\right),
\end{aligned}$$

given Equations (3)–(4). By differentiation with respect to $e > 0$, one has

$$\begin{aligned}
&f_{\mathcal{E}_t,\theta}\left(e \mid e_{t-1}\right) \\
&= f_{A_t,\theta}\left(\mathcal{E}_t^{-1}\left(e\right) \mid a_{t-1}\right) \times \mathcal{E}_t^{-1}\left(e\right)' \\
&= \frac{f_{A_t,\theta}\left(a \mid a_{t-1}\right)}{\mathcal{E}_t\left(a\right)'},
\end{aligned}$$

for $e > 0$ and $a = \mathcal{E}_t^{-1}\left(e\right) > b_t$. This results in the (straight) likelihood function

$$\mathcal{L}\left(\theta \mid e_0, \ldots, e_T\right) = \prod\nolimits_{t=1}^{T} \frac{f_{A_t,\theta}\left(a_t \mid a_{t-1}\right)}{\mathcal{E}_t\left(a_t\right)'}, \tag{5}$$

where $a_t = \mathcal{E}_t^{-1}\left(e_t\right) > b_t$ is the pseudo-observation associated with the observation $e_t > 0$, and $\mathcal{E}_t\left(a_t\right)'$ is the derivative of $\mathcal{E}_t$ evaluated at a_t, for $t \in \mathcal{P}$.

The likelihood expression in Equation (5) can be reworked otherwise if we consider the Gaussian Markov process $Z_t = \log\left(A_t\right)$ as a state variable. The same arguments give

$$\mathcal{L}\left(\theta \mid e_0, \ldots, e_T\right) = \prod\nolimits_{t=1}^{T} \frac{f_{Z_t,\theta}\left[z_t \mid z_{t-1}\right]}{a_t \times \mathcal{E}_t\left(a_t\right)'}, \tag{6}$$

where $z_0 = \log\left(a_0\right), \ldots, z_T = \log\left(a_T\right)$ are the pseudo-observations.

It is worth noticing that the joint distribution of A evaluated at $(a_0, \ldots, a_T)$, that is,

$$f_{A_0,\theta}\left(a_0\right) \times \prod\nolimits_{t=1}^{T} f_{A_t,\theta}\left(a_t \mid a_{t-1}\right),$$

cannot play the role of a likelihood function, as assumed by Duan (1994), since the pseudo-observations $a_0, \ldots, a_T$ are not (directly) observable. Duan (2000) reports a correction under Merton's setting. The general expressions in Equations (5)–(6) hold for all structural settings where the firm's asset value is the sole state variable.

The fact that we only observe surviving companies results in a lack of information under the default event, which results in a survival bias. The likelihood function can be adjusted consequently, as follows:

$$\begin{aligned}&\mathcal{L}\left(\theta \mid e_0,\ldots,e_T\right)\\ &=\prod\nolimits_{t=1}^{T} f_{\mathcal{E}_t,\theta}\left(e_t \mid e_{t-1} \text{ and } \mathcal{E}_t>0\right)\\ &=\prod\nolimits_{t=1}^{T} \frac{f_{\mathcal{E}_t,\theta}\left(e_t \mid e_{t-1}\right)}{\mathbb{P}\left(\mathcal{E}_t>0 \mid e_{t-1}\right)},\end{aligned}$$

which can be written as

$$\mathcal{L}\left(\theta \mid e_0,\ldots,e_T\right)=\prod\nolimits_{t=1}^{T} \frac{f_{A_t,\theta}\left(a_t \mid a_{t-1}\right)}{\mathbb{P}\left[A_t>b_t \mid a_{t-1}\right]\times \mathcal{E}_t\left(a_t\right)'}, \tag{7}$$

or, equivalently,

$$\mathcal{L}\left(\theta \mid e_0,\ldots,e_T\right)=\prod\nolimits_{t=1}^{T} \frac{f_{Z_t,\theta}\left[z_t \mid z_{t-1}\right]}{\mathbb{P}\left[A_t>b_t \mid a_{t-1}\right]\times a_t \times \mathcal{E}_t\left(a_t\right)'}. \tag{8}$$

2.3 Resolution

For a fixed value of θ, we exchange the value functions of corporate securities, the default barriers, and the pseudo-observations by their SDP counterparts, and compute the quasi-likelihood function $\widehat{\mathcal{L}}$. In particular, one has

$$\widehat{\mathcal{E}}_t\left(a_t\right)'=\beta_i, \quad \text{where } a_t\in\left(x_i,x_{i+1}\right],$$

given the SDP firm's equity value in Equation (2). The fact that $\widehat{\mathcal{E}}_t$ isn't continuous on the SDP evaluation nodes is not really a big issue since the a_t's and the x_i's almost surely don't match.

The rest consists of solving the likelihood equation

$$\widehat{\theta}^{\mathrm{QML}}=\arg\min_{\theta}-\log\widehat{\mathcal{L}}, \tag{9}$$

given the observed time series $e_0,\ldots,e_T$. This optimization problem is complex to achieve as $\widehat{\mathcal{L}}$ isn't an explicit expression of θ. We use the mesh adaptive direct search (MADS) algorithm of Audet and Dennis (2006) to achieve this task. Figure 1 shows the shape of $\widehat{\mathcal{L}}$ under the lognormal assumption.

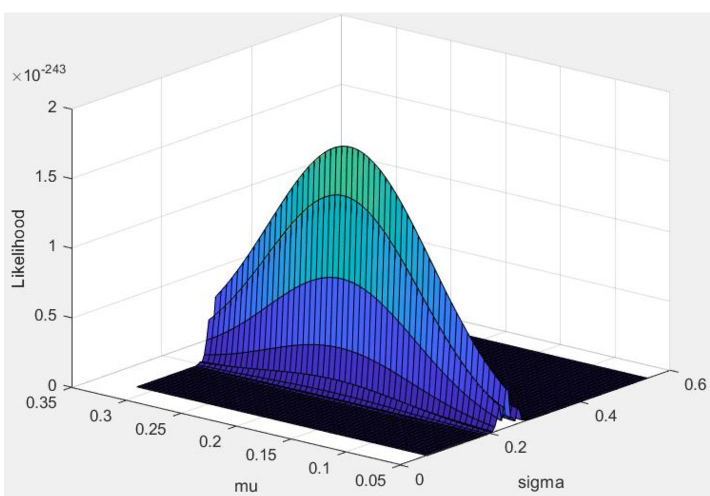

Figure 1: Curve of $\widehat{\mathcal{L}}$ under the lognormal hypothesis

Monte Carlo simulation can be used to estimate the variance matrix of $\widehat{\theta}^{\text{QML}}$ as follows:

1. Simulate a sample of N trajectories of the firm's asset value A over $[0, T]$, given the estimation $\widehat{\theta}^{\text{QML}}$ (as if it were the true value) of θ;
2. For each simulated path n of A, compute the QML estimate $\widehat{\theta}^{n}$ of θ, for $n = 1, \dots, N$;
3. Compute the sample variance of $\widehat{\theta}^{\text{QML}}$.

3 Empirical investigation

The code lines are written in C and compiled under CPP. We use the GSL and NOMAD software libraries to ease solving the model and searching for the QML estimates. See Le Digabel (2011) for further details on NOMAD.

To assess our QML approach, we use a Monte Carlo simulation under Merton's (1974) setting. We set the model's parameters as follows:

1. a drift and volatility of the underlying lognormal process $\mu_0 = 10\%$ (per year) and $\sigma_0 = 20\%$ (per year);
2. a risk-free rate $r = 5\%$ (per year);
3. a maturity and a principal amount of the pure corporate bond $T^D = 5$ years and $P = 100$ dollars;
4. a daily observation window $[0, T]$ of two years assuming 250 days per year.

We use the true value of θ, simulate a daily path of A over $[0, T]$ under the physical probability measure, and provide the QML estimate of θ. Then, we perform $N = 2000$ replications of the same experiment and compute the root mean square error (RMSE) of ML vs QML estimates of θ.

Table 1: RMSE of ML vs QML estimates of θ – Simulated model

Estimation	Search method	RMSE($\widehat{\mu}$)	RMSE($\widehat{\sigma}$)
ML (Duan)	Newton-based	10.4%	4.5%
QML	Naive search	14.1%	5.1%
QML	MADS algorithm	12.9%	3.4%

ML and QML estimates of θ are comparable and show that the estimation of the volatility parameter is more accurate than of the drift parameter, as documented in the literature under the lognormal assumption.

We now conduct a credit-risk analysis of Phar-Mor, a public chain of discount drug stores. Phar-Mor went bankrupt in September 2001, while its debt maturity T^D was in March 2002. We set the present date T at one year then one year and a half before bankruptcy, and we observe the stock closing price over the 2-years time window $[0, T]$. The model respects the actual firm's interest and capital payment schedule over $\left[0, T^D\right]$. The risk-free rate r is taken as the yield to maturity of a Treasury pure bond portfolio with an identical payment schedule to that of Phar-Mor.

We assume a Geske-like (1977) setting with TB $= 0$, BC $= 0$, and $D^j = 0$. Table 2 reports the QML estimates of $\theta = (\mu, \sigma)'$, the drift and diffusion parameters of the lognormal state process A.

Table 2: QML estimate of θ – Geske (1977) setting

Present date T	$\widehat{\mu}^{\text{QML}}$	$\widehat{\sigma}^{\text{QML}}$	r
12 months before bankruptcy	-27.5%	29.0%	5.52%
18 months before bankruptcy	-9.3%	28.6%	5.53%

Table 3 exhibits the yield to maturity (YTM) of Phar-Mor's debt portfolio, its yield spread (YS), and its term structure of total default probabilities (TDP) unde the physical probability measure for the horizons 6, 12, and 18 months at the present date T.

Table 3: YTM, YS, and TDP – Geske (1977) setting

Present date T	YTM	YS	TDP6	TDP12	TDP18
12 months before bankruptcy	8.24%	2.73%	26%	48%	70%
18 months before bankruptcy	7.31%	1.78%	4%	11%	20%

Credit-risk indicators deteriorate when we approach bankruptcy, but those computed under the physical probability measure are more sensitive. This contradicts Delianedis and Geske (2003) who focus on changes in risk-neutral credit-risk measures. In line with the literature on the structural model and the credit-spread puzzle, Phar-Mor's yield spreads remain largely under-valued. Meanwhile, Phar-Mor was rated B3 from November 1995 to February 2001, as reported in Moody's Default & Recovery Data Base. See Collin–Dufresne et al. (2001), Driessen (2005), Chen (2010), Bao et al. (2011), Gemmill and Keswani (2011), Huang and Huang (2012), and Du et al. (2019) for further details on the credit-spread puzzle.

We adjust the likelihood function for the survival bias in an attempt to revisit Phar-Mor's credit spreads. We observe only a few effects on the QML estimates of $\theta = (\mu, \sigma)'$ and the credit-risk measures. This finding is consistent with Duan et al. (2004) and Amaya et al. (2019).

We now consider the structural model with frictions in Equation (1) to align Phar-Mor credit spreads in Table 3 with Moody's B3 rating. We fix the corporate tax rate at 35% and vary w from 20% to 60%. ML and QML based on stock prices present a common weak point as they don't allow for the estimation of the bankruptcy-cost parameter w.

Table 4: QML estimates of θ – Extended Geske (1977) setting

Present date T	w	$\widehat{\mu}^{\text{QML}}$	$\widehat{\sigma}^{\text{QML}}$
12 months before bankruptcy	20%	-26.1%	31.0%
18 months before bankruptcy	20%	-10.0%	30.0%
12 months before bankruptcy	40%	-26.5%	31.0%
18 months before bankruptcy	40%	-10.0%	30.0%
12 months before bankruptcy	60%	-26.5%	31.0%
18 months before bankruptcy	60%	-9.1%	29.8%

In contrast to default probabilities, yield spreads are sensitive to w, as shown in Table 5.

Table 5: YTM, YS, and TDP – Extended Geske (1977) setting

Present date T	w	YTM	YS	TDP6	TDP12	TDP18
12 months before bankruptcy	20%	10.42%	4.91%	22%	47%	74%
18 months before bankruptcy	20%	8.37%	2.84%	3%	10%	20%
12 months before bankruptcy	40%	12.08%	6.57%	22%	48%	74%
18 months before bankruptcy	40%	9.12%	3.59%	3%	10%	20%
12 months before bankruptcy	60%	13.80%	8.29%	22%	48%	74%
18 months before bankruptcy	60%	9.78%	4.25%	2%	9%	19%

A bankruptcy-cost parameter w of 40% results in yield spreads that are likely to match Moody's rating.

4 Conclusion

We propose a flexible QML approach to estimate complex structural settings. We conduct an empirical investigation of a public company that went bankrupt. We show that its credit-risk indicators deteriorate when we approach bankruptcy, while its yield spreads remain under-valued. This is consistent with the literature on structural models and the credit-spread puzzle. We consider bankruptcy costs as a remedy.

A future research avenue consists of adjusting QML to estimate the bankruptcy-cost parameter(s) together with the state-process parameters.

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