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Estimation and goodness-of-fit for regimes-switching copula models with application

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Abstract: We consider several time series and for each of them, we fit an appropriate dynamic parametric model. This produces serially independent error terms for each time series. The dependence between these error terms is then modeled by a regime-switching copula. The EM algorithm is used for estimating the parameters and a sequential goodness-of-fit procedure based on Cramér-von Mises statistics is proposed to select the appropriate number of regimes. Numerical experiments are performed to assess the validity of the proposed methodology. As an example of application, we evaluate a European put-on-max option on the returns of two assets. In order to facilitate the use of our methodology, we have built a R package HMMcopula available on CRAN.

Keywords: Goodness-of-fit, time series, copulas, regime-switching models, generalized error models

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1 Introduction

In finance, many instruments are based on several risky assets and their evaluation rest on the joint distribution of these assets. In fact, to determine this joint distribution, we must take into account the serial dependence in each asset, as well as the dependence between the assets. Underestimating the latter can have devastating financial and economic consequences, as exemplified by the 2008 financial crisis. We must also consider that the dependence may vary with time, potentially increasing in crisis periods. Some ways to take into account time-varying dependence have been proposed. Recently, Adams et al. (2017) fitted DCC-GARCH models (Engle, 2002) to multivariate time series, which is a bit restrictive in terms of dependence since it is based on the multivariate Gaussian distribution. To overcome this limitation, and because copulas are specially designed to model dependence, it is no wonder that many time-varying dependence models are based on copulas.

To our knowledge, one of the first papers involving time-dependent copulas was van den Goorbergh et al. (2005), where the authors, in order to evaluate call-on-max options, fitted a copula family to the residuals of two GARCH time series, with a parameter expressed as a function of the volatilities. Note that this is a special case of what is now known as single-index copula (Fermanian and Lopez, 2018). One can also use the methodology proposed in Nasri and Rémillard (2019), where generalized error models are fitted to each time series, and the underlying copula has time-dependent parameters. In order to be able to take into account abrupt changes in the dependence, it can be appropriate to use regime-switching copulas. This approach has been proposed recently by Fink et al. (2017), who fitted their model to residuals of univariate GARCH series. They did not propose a formal test of goodness-of-fit, and the selection of the number of regimes was based on comparisons of likelihoods, which is not recommended (Cappé et al., 2005) as a selection method, due to possible singularities and non-identifiable parameters. Here, we propose a formal goodness-of-fit test for regime-switching copulas, which was not done before in this setting. As a by-product, we obtain an interesting way to select the number of regimes.

More precisely, in Section 2, we describe the model for the time series and we define regime-switching copulas. In Section 3, we detail the estimation procedure, the goodness-of-fit test and the selection of the number of regimes. Numerical experiments are performed in Section 4 to assess the validity of the proposed methodology to choose the number of regimes. In Section 5, we give an example of application for option pricing, along the same lines as van den Goorbergh et al. (2005) but with different data.

2 Linking multivariate time series with regime-switching copulas

To introduce copula-based models, we proceed in two steps: first, for each univariate time series, we use a "generalized error model" (Du, 2016) to produce iid univariate series; second, regime-switching copulas are fitted to these series. To fix ideas, let $\mathbf{X}_t = (X_{1t}, \ldots, X_{dt})$, be a multivariate time series. For each $j \in \{1, \ldots, d\}$, let $\mathcal{F}_{j,t-1}$ contains information from the past of $X_{j1}, \ldots X_{j,t-1}$, and possibly information from exogenous variables as well. Further set $\mathcal{F}_t = \vee_{j=1}^d \mathcal{F}_{j,t}$. Assume that for each $j \in \{1, \ldots, d\}$, there exist continuous, increasing, and $\mathcal{F}_{j,t-1}$-measurable functions $G_{\boldsymbol{\alpha},jt}$ so that $\varepsilon_{jt} = G_{\boldsymbol{\alpha},jt}(X_{jt})$ are iid with continuous distribution function F_j and density f_j, for some unknown parameter $\boldsymbol{\alpha} \in \mathcal{A}$. Note that stochastic volatility models and Hidden Markov models (HMM) are particular cases of generalized error models. Next, to introduce the dependence between the time series, we choose a sequence of $\mathcal{F}_{t-1}$-measurable copulas C_t, so that the joint conditional distribution function K_t of $\boldsymbol{\varepsilon}_t = (\varepsilon_{1t}, \ldots, \varepsilon_{dt})$ given $\mathcal{F}_{t-1}$ is $K_t(\mathbf{x}) = C_t\{\mathbf{F}(\mathbf{x})\}$, with $\mathbf{F}(\mathbf{x}) = (F_1(x_1), \ldots, F_d(x_d))^\top$, for any $\mathbf{x} = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d$. In particular, $\mathbf{U}_t = \mathbf{F}(\boldsymbol{\varepsilon}_t) \sim C_t$, for every $t \in \{1, \ldots, n\}$.

This way of modeling dependence between several time series is usually applied to innovations of stochastic volatility models (van den Goorbergh et al., 2005, Chen and Fan, 2006, Patton, 2006, Rémillard, 2017). For example, suppose that $X_{1t} = \mu_{1t}(\boldsymbol{\alpha}) + \sigma_{1t}(\boldsymbol{\alpha})\varepsilon_{1t}$, $\varepsilon_{1t} \sim F_1$, where μ_{1t} and σ_{1t} are $\mathcal{F}_{1,t-1}$-measurable, and the innovations ε_{1t} are independent of $\mathcal{F}_{1,t-1}$. In this case, one could take $G_{\boldsymbol{\alpha},1t}(x_1) = \frac{x_1-\mu_{1t}(\boldsymbol{\alpha})}{\sigma_{1t}(\boldsymbol{\alpha})}$, and then $\varepsilon_{1t} = G_{\boldsymbol{\alpha},1t}(X_{1t}) \sim F_1$. We can also consider Gaussian HMM models for some univariate time series. For example, suppose that there exists a Markov chain s_t on $\{1,\ldots,m\}$ with transition matrix Q so that given $s_1 = i_1,\ldots,s_n = i_n$, $X_{11},\ldots,X_{1n}$ are independent, and $X_{1t} \sim N(\mu_{i_t}, \sigma^2_{it})$. If $\eta_{t-1}(k)$ is the probability of being in regime $k \in \{1,\ldots,m\}$ at time $t-1$ given the past observations $X_{11},\ldots,X_{1,t-1}$, then the conditional distribution G_{1t} of X_{1t} given the past is $G_{1t}(x) = \sum_{k=1}^m W_{t-1}(k)F^{(k)}(x)$, where $W_{t-1}(k) = \sum_{j=1}^m \eta_{t-1}(j)Q_{jk}$, $\eta_{t-1}(j)$ is the probability of being in regime j at time $t-1$ given the past observations, and $F^{(k)}$ is the cdf of a Gaussian distribution with mean μ_k and variance σ^2_k. It then follows that the sequence $U_{1t} = G_{1t}(X_{1t})$ are iid uniform random variables.

After having chosen the generalized error models for each univariate time series, we need to choose the regime-switching copula model C_t for the multivariate series $\mathbf{U}_t$. This means that there exists a finite Markov chain τ_t on $\{1,\ldots,\ell\}$ with transition matrix P, so that given $\tau_1 = i_1,\ldots,\tau_n = i_n$, $\mathbf{U}_1,\ldots,\mathbf{U}_n$ are independent, and $\mathbf{U}_t \sim C_{\boldsymbol{\beta}_{i_t}}$, $t \in \{1,\ldots,n\}$, where $\{C_{\boldsymbol{\beta}}; \boldsymbol{\beta} \in \mathcal{B}\}$, is a given parametric copula family. Also we assume the usual smoothness conditions on the associated densities $c_{\boldsymbol{\beta}}$ so that the pseudo-maximum likelihood estimator exists. Note that for a given $j \in \{1,\ldots,d\}$, one needs that the values U_{jt}, $t \in \{1,\ldots,n\}$, are iid uniform. This is indeed true as proven in the following theorem.

Theorem 1 *Suppose that the multivariate time series $\mathbf{U}_t$ has distribution function C_t given $\mathcal{F}_{t-1}$. Then for any given $j \in \{1,\ldots,d\}$, the values U_{jt}, $t \in \{1,\ldots,n\}$, are iid uniform.*

Proof. For simplicity, suppose that $j = 1$. By hypothesis, $\mathbb{P}(U_{1t} \le u_1,\ldots,U_{dt} \le u_d|\mathcal{F}_{t-1}) = C_t(u_1,\ldots,u_d)$. From the properties of copulas, one gets that $\mathbb{P}(U_{1t} \le u_1|\mathcal{F}_{t-1}) = C_t(u_1,1,\ldots,1) = u_1$. As a result, one may conclude that $U_{11},\ldots,U_{1n}$ are iid uniform. □

Since the generalized errors $\boldsymbol{\varepsilon}_t$ are not observable, $\boldsymbol{\alpha}$ being unknown, the latter must be estimated by a consistent estimator $\boldsymbol{\alpha}_n$. One can then compute the pseudo-observations $\mathbf{e}_{n,t} = (e_{n,1t},\ldots,e_{n,dt})^\top = \mathbf{G}_{\boldsymbol{\alpha}_n,t}(\mathbf{X}_t)$, where $e_{n,jt} = G_{\boldsymbol{\alpha}_n,jt}(X_{jt})$, $j \in \{1,\ldots,d\}$ and $t \in \{1,\ldots,n\}$. Using these pseudo-observations might be a problem, but in Nasri and Rémillard (2019), it was shown that using the normalized ranks of these pseudo-observations, one can estimate the parameters $\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_\ell$ and P, as if one was observing $\mathbf{U}_1,\ldots,\mathbf{U}_n$. The same applies to the goodness-of-fit test that will be defined in Section 3.2.

Based on Theorem 1, note that in order to simulate the multivariate time series, it suffices to generate $\mathbf{U}_t = (U_{1t},\ldots,U_{dt})$ according to the regime-switching copula model, set $\varepsilon_{jt} = F_j^{-1}(U_{jt})$, and then compute $X_{jt} = G^{-1}_{\boldsymbol{\alpha},jt}(\varepsilon_{jt})$, $j \in \{1,\ldots,d\}$, and $t \in \{1,\ldots,n\}$.

3 Estimation and goodness-of-fit test

We first present general regime-switching models which can be applied to univariate time series or copula. Then, we describe an estimation procedure and a goodness-of fit test for regime-switching copula models. Finally, we propose a sequential procedure for selecting the optimal number of regimes.

3.1 General regime-switching models

Let τ_t be a homogeneous discrete-time Markov chain on $S = \{1,\ldots,\ell\}$, with transition probability matrix P on $S \times S$. Given $\tau_1 = k_1,\ldots,\tau_n = k_n$, the observations $\mathbf{Y} = (Y_1,\ldots,Y_n)$ are independent with densities $g_{\boldsymbol{\beta}_{k_t}}$, $t \in \{1,\ldots,n\}$. Set $\boldsymbol{\theta} = (\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_\ell, P)$. Then the joint density of $\tau = (\tau_1,\ldots,\tau_n)$ and $\mathbf{Y}$ is

$$f_{\boldsymbol{\theta}}(\tau,\mathbf{Y}) = \left(\prod_{t=1}^n P_{\tau_{t-1},\tau_t}\right) \times \prod_{t=1}^n g_{\boldsymbol{\beta}_{\tau_t}}(Y_t), \tag{1}$$

so one can write

$$\log f_{\boldsymbol{\theta}}(\tau, \mathbf{Y}) = \sum_{t=1}^{n} \log P_{\tau_{t-1},\tau_t} + \sum_{t=1}^{n} \log g_{\boldsymbol{\beta}_{\tau_t}}(Y_t). \tag{2}$$

Because the regimes τ_t are not observable, an easy way to estimate the parameters is to use the EM algorithm (Dempster et al., 1977), which proceeds in two steps: expectation (E step), where $Q_{\mathbf{y}}(\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta}) = \mathbb{E}_{\boldsymbol{\theta}}\{\log f_{\tilde{\boldsymbol{\theta}}}(\tau, \mathbf{Y}) | \mathbf{Y} = \mathbf{y}\}$ is computed, and maximization (M step), where one computes

$$\boldsymbol{\theta}^{(k+1)} = \arg\max_{\boldsymbol{\theta}} Q_y\left(\boldsymbol{\theta}, \boldsymbol{\theta}^{(k)}\right),$$

starting from an initial value $\boldsymbol{\theta}^{(0)}$. As $k \to \infty$, $\boldsymbol{\theta}^{(k)}$ converges to the maximum likelihood estimator of the density of $\mathbf{Y}$. The formulas for the EM steps are given in Appendix A. As a particular case of regime-switching models, if $P_{ij} = \nu_j$, then one gets mixture models. In this case $\tau_1, \ldots, \tau_n$ are iid The simplified formulas for the EM steps are given in Appendix B. For application to copulas, the density $g_{\boldsymbol{\beta}}$ is the density of a parametric family of copulas $C_{\boldsymbol{\beta}}$, with $\boldsymbol{\beta} \in \mathcal{B}$. However $Y_1, \ldots, Y_n$ are not observable so they must be replaced by the normalized ranks of the pseudo-observations $\mathbf{e}_{n,t}$, i.e., $Y_{jt} = \text{rank}(e_{n,jt})/(n+1)$.

3.2 Goodness-of-fit

In this section, we propose a methodology to perform a goodness-of-fit test on a multivariate time series, by using the Rosenblatt's transform. First, following Rémillard (2013), under the general regime-switching model described in Section 3.1, the conditional density f_t of Y_t given $Y_1, \ldots, Y_{t-1}$ can be expressed as a mixture viz.

$$f_t(y_t | y_1, \ldots, y_{t-1}) = \sum_{i=1}^{\ell} f^{(i)}(y_t) \sum_{j=1}^{\ell} \eta_{t-1}(j) P_{ji} = \sum_{i=1}^{\ell} f^{(i)}(y_t) W_{t-1}(i), \tag{3}$$

where $f^{(i)} = g_{\boldsymbol{\beta}_i}$ and

$$\begin{aligned} W_{t-1}(i) &= \sum_{j=1}^{\ell} \eta_{t-1}(j) P_{ji}, \quad i \in \{1 \ldots \ell\}, && (4) \\ \eta_t(j) &= \frac{f^{(j)}(y_t)}{Z_{t|t-1}} \sum_{i=1}^{\ell} \eta_{t-1}(i) P_{ij}, \quad j \in \{1, \ldots, \ell\}, && (5) \\ Z_{t|t-1} &= \sum_{j=1}^{\ell} f^{(j)}(y_t) \sum_{i=1}^{\ell} \eta_{t-1}(i) P_{ij}. && (6) \end{aligned}$$

Note that formulas (3)–(6) also hold for univariate Gaussian HMM; in this case, $f^{(j)}$ is the Gaussian density with mean μ_j and variance σ_j^2. Next, let $i \in \{1, \ldots, \ell\}$ be fixed and suppose that $Z = (Z_1, \ldots, Z_d)$ has density $f^{(i)}$. For any $q \in \{1, \ldots, d\}$, denote by $f_{1:q}^{(i)}$ the density of $(Z_1, \ldots, Z_q)$. Also, let $f_q^{(i)}$ be the conditional density of Z_q given $Z_1, \ldots, Z_{q-1}$. Further denote by $F_q^{(i)}$ the distribution function corresponding to density $f_q^{(i)}$. The Rosenblatt's transform Ψ_t corresponding to the density (3) conditional on $y_1, \ldots, y_{t-1} \in \mathbb{R}^d$ is given by

$$\Psi_t^{(1)}(y_{1t}) = \sum_{i=1}^{\ell} W_{t-1}(i) F_1^{(i)}(y_{1t}), \tag{7}$$

and for $q \in \{2, \ldots, d\}$,

$$\Psi_t^{(q)}(y_{1t}, \ldots, y_{qt}) = \frac{\sum_{i=1}^{\ell} W_{t-1}(i) f_{1:q-1}^{(i)}(y_{1t}, \ldots, y_{q-1,t}) F_q^{(i)}(y_{tq})}{\sum_{i=1}^{\ell} W_{t-1}(i) f_{1:q-1}^{(i)}(y_{1t}, \ldots, y_{q-1,t})}. \tag{8}$$

Suppose now that $\mathbf{U}_1, \ldots, \mathbf{U}_n$ is a random sample of size n of d-dimensional vectors drawn from a joint continuous distribution $\mathbf{P}$ belonging to a parametric family of regime-switching copula models with ℓ regimes. Formally, the hypothesis to be tested is

$$\mathcal{H}_0 : \mathbf{P} \in \mathcal{P} = \{\mathbf{P}_{\boldsymbol{\theta}}; \boldsymbol{\theta} \in \mathcal{O}\} \quad vs \quad \mathcal{H}_1 : \mathbf{P} \notin \mathcal{P}.$$

Under $\mathcal{H}_0$, it follows that $\mathbf{V}_1 = \Psi_1(\mathbf{U}_1, \boldsymbol{\theta}), \mathbf{V}_2 = \Psi_2(\mathbf{U}_1, \mathbf{U}_2, \boldsymbol{\theta}), \ldots, \mathbf{V}_n = \Psi_n(\mathbf{U}_1, \ldots, \mathbf{U}_n, \boldsymbol{\theta})$ are iid uniform over $(0,1)^d$, where $\Psi_1(\cdot, \boldsymbol{\theta}), \ldots, \Psi_n(\cdot, \boldsymbol{\theta})$ are the Rosenblatt's transforms for the true parameters $\boldsymbol{\theta} \in \mathcal{O}$. However, $\boldsymbol{\theta}$ must be estimated, say by $\boldsymbol{\theta}_n$. Also, the random vectors $\mathbf{U}_1, \ldots, \mathbf{U}_n$ are not observable, so they must be replaced by the normalized ranks $\mathbf{u}_{n,t}$ of the pseudo-observations $\mathbf{e}_{n,t}$, $t \in \{1, \ldots, n\}$. Then, define the pseudo-observations $\mathbf{V}_{n,t} = \Psi_t(\mathbf{u}_{n,t}, \boldsymbol{\theta}_n)$, $t \in \{1, \ldots, n\}$, and for any $\mathbf{u} = (u_1, \ldots, u_d) \in [0,1]^d$, define the empirical process $D_n(\mathbf{u}) = \frac{1}{n} \sum_{t=1}^n \prod_{j=1}^d \mathbb{1}\left(V_{n,jt} \leq u_j\right)$. To test $\mathcal{H}_0$ against $\mathcal{H}_1$, Genest et al. (2009) suggest to use the Cramér-von Mises type statistic S_n defined by

$$\begin{aligned} S_n &= B_n(\mathbf{V}_{n,1}, \ldots, \mathbf{V}_{n,n}) = n \int_{[0,1]^d} \left\{ D_n(\mathbf{u}) - \prod_{j=1}^d u_j \right\}^2 d\mathbf{u} \\ &= \frac{1}{n} \sum_{t=1}^n \sum_{i=1}^n \prod_{q=1}^d \left\{1 - \max\left(V_{n,qt}, V_{n,qi}\right)\right\} - \frac{1}{2^{d-1}} \sum_{t=1}^n \prod_{q=1}^d \left(1 - V_{n,qt}^2\right) + \frac{n}{3^d}. \end{aligned}$$

We can interpret S_n as the distance of our empirical distribution and the independence copula. Since $\mathbf{V}_{n,t}$, $t \in \{1, \ldots, n\}$ are almost uniformly distributed over $[0,1]^d$ under $\mathcal{H}_0$, large values of S_n lead to the rejection of the null hypothesis. Unfortunately, the limiting distribution of the test statistic will depend on the unknown parameter $\boldsymbol{\theta}$, but it does not depend on the estimated parameters of the univariate time series (Nasri and Rémillard, 2019). Therefore, we will use the parametric bootstrap described in Algorithm 1 to estimate P-values.

Algorithm 1 *For a given number of regimes ℓ, get estimator $\boldsymbol{\theta}_n$ of $\boldsymbol{\theta}$ using the EM algorithm described in Section 3.1, applied to the pseudo-observations $\mathbf{u}_{n,t}$, $t \in \{1, \ldots, n\}$. Then compute the statistic $S_n = B_n(\mathbf{V}_{n,1}, \ldots, \mathbf{V}_{n,n})$, using the pseudo-observations $\mathbf{V}_{n,t} = \Psi_t(\mathbf{u}_{n,t}, \boldsymbol{\theta}_n)$, $t \in \{1, \ldots, n\}$. Then for $k = 1, \ldots, B$, B large enough, repeat the following steps:*

- *Generate a random sample $\mathbf{U}_1^*, \ldots, \mathbf{U}_n^*$ from distribution $\boldsymbol{P}_{\boldsymbol{\theta}_n}$, i.e., from a regime-switching copula model with parameter $\boldsymbol{\theta}_n$.*
- *Get the estimator $\boldsymbol{\theta}_n^*$ from $\mathbf{U}_1^*, \ldots, \mathbf{U}_n^*$.*
- *Compute the normalized ranks $\mathbf{u}_{n,1}^*, \ldots, \mathbf{u}_{n,n}^*$ from $\mathbf{U}_1^*, \ldots, \mathbf{U}_n^*$.*
- *Compute the pseudo-observations $\mathbf{V}_{n,t}^* = \boldsymbol{\Psi}_t\left(\mathbf{u}_{n,t}^*, \boldsymbol{\theta}_n^*\right)$, $t \in \{1, \ldots, n\}$ and calculate $S_n^{(k)} = B_n\left(\mathbf{V}_{n,1}^*, \ldots, \mathbf{V}_{n,n}^*\right)$.*

Then, an approximate P-value for the test based on Cramér-von Mises statistic S_n is given by

$$\frac{1}{B} \sum_{k=1}^B \mathbb{1}\left(S_n^{(k)} > S_n\right).$$

As suggested in Rémillard et al. (2017), it makes sense to choose the number of regimes ℓ^* as the first ℓ for which the P-value is larger than 5%. This procedure is investigated numerically next.

4 Numerical experiments

In this section we consider Monte Carlo experiments for assessing the validity of the proposed sequential procedure to estimate the number of regimes, i.e., we choose $\ell^\star$ to be the smallest number of regimes (starting at 1) for which the P-value is greater or equal than 5%. To this end, we generated random samples of size $n \in \{250, 500, 1000\}$ from four regime-switching bivariate copula families: Clayton, Frank, Gaussian, and Gumbel with 1 and 2 regimes. For the 1-regime model, all copulas have a Kendall's $\tau = .5$, while for the 2-regime copula, we took $\tau = .25$ for regime 1 and $\tau = 0.75$ for regime 2, with transition matrix $P = \begin{pmatrix} 0.25 & 0.75 \\ 0.50 & 0.50 \end{pmatrix}$. For each sample size, we performed 1000 replications and in each replication, $B = 100$ bootstrap samples were used to compute the P-value of the test statistic S_n. The results of these Monte Carlo experiments are displayed in Table 1. We can see from Table 1 that the proposed methodology works fine, especially when the sample size is large enough. In fact, in the case of 1 regime, one should expect to get $\ell^\star = 1$, 95% of the time, which is true for all tested models. However, in the case of 2 regimes, one cannot expect to get $\ell^\star = 2$, 95% of the time, since one starts by testing 1 regime and if the P-value is larger or equal to 5%, we do not compute the P-value for 2 regimes. Nevertheless, the results are quite satisfactory especially for the Clayton and Gaussian families where we find the right number of regimes more than 88% of the time, even with $n = 250$, while for the Frank and Gumbel families, the results are good enough when $n \geq 500$. One should also expect better results by taking a larger number B of bootstrap samples. Here, in order to build the tables in a reasonable time, we restricted ourself to $B = 100$, which is quite small. In real life, we do not repeat the experiments 1000 times, so we may use at least $B = 1000$, especially when the P-value is around 5%. Furthermore, we did not consider the regime-switching Student copula since it has more parameters and, according to Table 4, the computation time is approximatively 10 times longer for the Student family than for the Gumbel family, which has the longest computation time amongst the four other studied families.

Table 1: Estimation of the number of regimes ℓ^* for $N = 1000$ experiments, using $B = 100$ bootstrap samples. Boldface values indicate the percentage of the correct choice of the number of regimes.

	Copula family							
	Clayton		Frank		Gaussian		Gumbel	
	Number of regimes		Number of regimes		Number of regimes		Number of regimes	
$\ell^\star$	1	2	1	2	1	2	1	2
				$n = 250$				
1	**94.4**	0.8	**94.8**	25.1	**95.3**	5.4	**95.2**	37.5
2	2.3	**91.8**	1.5	**57.0**	1.6	**88.7**	1.5	**58.0**
3	0.5	2.9	0.7	2.3	0.4	2.3	0.4	1.2
≥ 4	2.8	4.5	3.0	15.6	2.7	3.6	3.6	3.3
				$n = 500$				
1	**93.8**	0	**94.8**	2.4	**93.6**	0.1	**95.2**	8.6
2	2.4	**92.3**	1.6	**76.4**	1.9	**95.4**	1.2	**86.7**
3	0.4	1.8	0.7	2.9	0.5	1.6	0.6	1.0
≥ 4	3.4	5.9	2.9	18.3	4	2.9	3.0	3.7
				$n = 1000$				
1	**95.0**	0.0	**94.1**	0.0	**95.6**	0.0	**94.9**	0.1
2	2.2	**94.3**	1.7	**79.5**	0.8	**95.2**	1.2	**93.7**
3	0.6	1.7	0.3	2.1	0.7	1.0	0.4	1.0
≥ 4	2.2	4.0	3.9	18.4	2.9	3.8	3.5	5.2

Finally, for sake of comparison, we also considered a choice of the number of regimes based on the AIC criterion, i.e., we computed the AIC for $\ell \in \{1, 2, 3, 4\}$ regimes and we selected the one yielding the smallest value. Note that in this case, one does not know if the model is valid. One just chooses the one that is the best model amongst the four proposed ones. These results are displayed in Table 2. From this table, one can see that the results are comparable to the ones using the P-value, especially when $n \geq 500$. However, for $n = 250$ the results obtained using the AIC are better for Frank and Gumbel families. As a general rule, one could start with the AIC criterion for a given family, then use the P-value of the statistic S_n to confirm that the model is valid.

Table 2: Choice of the number of regime ℓ^* for $N = 1000$ experiments, using the AIC criterion. Boldface values indicate the percentage of the correct number of regimes.

	Copula family							
	Clayton		Frank		Gaussian		Gumbel	
	Number of regimes		Number of regimes		Number of regimes		Number of regimes	
ℓ^*	1	2	1	2	1	2	1	2
	n = 250							
1	**94.3**	0.1	**95.3**	0.6	**95.6**	0.0	**94.4**	0.6
2	5.1	**91.7**	4.7	**95.6**	4.3	**95.5**	5.3	**95.7**
3	0.1	4.8	0.0	3.7	0.1	4.5	0.3	3.7
4	0.5	3.4	0.0	0.1	0.0	0.0	0.0	0.0
	n = 500							
1	**95.1**	0.0	**96.1**	0.0	**97.2**	0.0	**95.7**	0.0
2	4.5	**94.4**	3.9	**97.3**	2.8	**96.2**	4	**96.4**
3	0.2	5.4	0	2.6	0.0	3.8	0.3	3.6
4	0.2	0.2	0	0.1	0.0	0.0	0.0	0.0
	n = 1000							
1	**97.0**	0.1	**98.1**	0.0	**98.3**	0.0	**95.7**	0.0
2	2.9	**95.7**	1.9	**97.5**	1.7	**98.0**	4.3	**97.3**
3	0.1	4.2	0.0	2.5	0.0	2.0	0.0	2.7
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

5 Application to option pricing

In this application, we want to evaluate a European put-on-max option on Amazon (amzn) and Apple (aapl) stocks. The payoff of this option is given by $\Phi(s_1, s_2) = \max\{K - \max(s_1, s_2), 0\}$, where s_1 and s_2 are the values of the stocks at the maturity of the option, normalized to start at 1\$, and K is the strike price. In order to do this, we need first to find the joint distribution of both assets. Next, we will choose an appropriate risk neutral probability measure.

5.1 Joint distribution

The first step is to fit dynamic models for the univariate time series. To this end, we used the adjusted prices of Amazon and Apple from January 1, 2015 to June 29, 2018. The sample size is 881 observations for each time series. The daily log-returns of the stocks are shown in Figure 1. Since van den Goorbergh et al. (2005) used GARCH models for the log-returns of the assets they considered, we also tried to fit GARCH(p,q) models with Gaussian innovations, but we rejected the null hypothesis for $p, q \leq 3$. We next tried to fit HMM Gaussian models to the log-returns. Using the selection procedure described in Section 3.2, we obtained a 3-regime Gaussian HMM for the daily log-returns of Amazon as well as for the daily log-returns of Apple. Here, the P-values are 38.8% and 15.1% respectively, computed using $B = 1000$ bootstrap samples. The estimated parameters for both time series are given in Table 3.

Table 3: Estimated parameters for the log-returns of Amazon and Apple, using Gaussian HMM. Here, ν is the stationary distribution of the regimes.

	Amazon			Apple		
	Regime			Regime		
Parameter	1	2	3	1	2	3
$\mu \times 10^{-2}$	4.41	0.19	-0.12	0.043277	-0.28	0.22
$\sigma \times 10^{-3}$	3.3	0.10268	0.52519	0.42624	0.0061731	0.077751
ν	0.0199	0.7229	0.2572	0.3892	0.1015	0.5092
Q	$\begin{pmatrix} 0.1572 & 0.4450 & 0.3978 \\ 0.0038 & 0.9662 & 0.03 \\ 0.0545 & 0.0606 & 0.8849 \end{pmatrix}$			$\begin{pmatrix} 0.8788 & 0.0001 & 0.1211 \\ 0.4154 & 0.0674 & 0.5172 \\ 0.0098 & 0.1859 & 0.8043 \end{pmatrix}$		

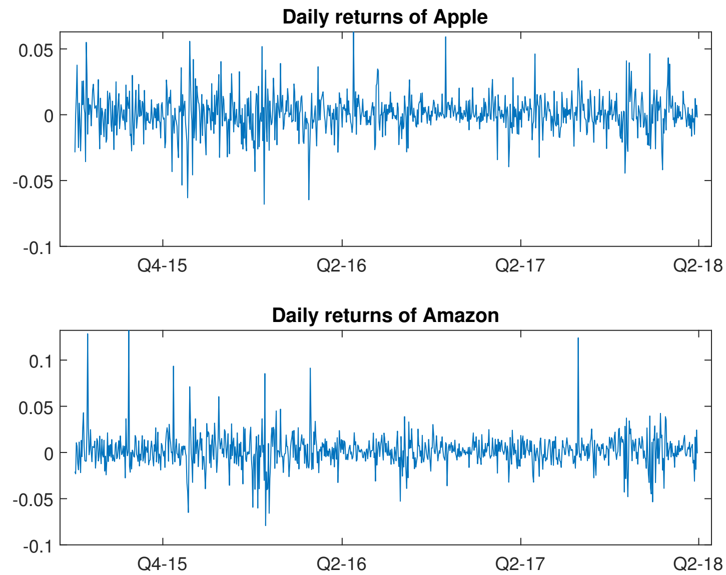

Figure 1: Daily log returns of Apple and Amazon.

From now on, let X_{1t} denotes the log-returns of Amazon and let X_{2t} denotes the log-returns of Apple, and let F_{1t} and F_{2t} be the conditional distributions of X_{1t} and X_{2t} given the past observations, corresponding to the densities defined by Equation (3). Further set $U_{1t} = F_{1t}(X_{1t})$ and $U_{2t} = F_{2t}(X_{2t})$. As defined in Section 3, let $u_{n,jt} = F_{n,jt}(X_{jt})$, $j = 1, 2$, be the pseudo-observations, where $F_{n,jt}$ is the conditional distribution function computed with the parameters of Table 3. The graph of the normalized ranks of $\mathbf{u}_{n,t} = (u_{n,1t}, u_{n,2t})$ is displayed in Figure 2. Next, in order to select the appropriate

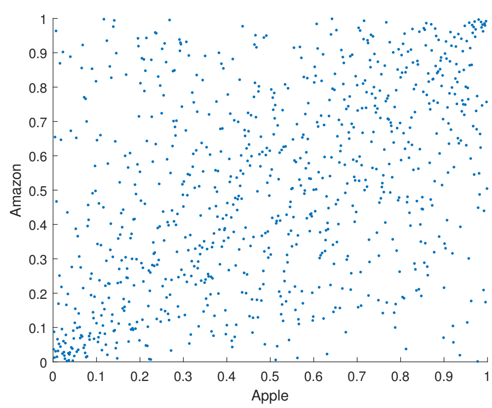

Figure 2: Scatter plot of the normalized ranks of the pseudo-observations $\mathbf{u}_{n,t}$ for Apple and Amazon.

regime-switching copula model, we performed goodness-of-fit tests using $B = 1000$ bootstrap samples to select the copula family and the number of regimes amongst the Clayton, Frank, Gaussian, Gumbel and Student families. The corresponding P-values are given in Table 4, together with the computation time in seconds. From this table, we can see that 2-regime Gaussian and Student copula models are valid. However, since the regime-switching Gaussian copula model has fewer parameters and the P-values are closed, we opted for the regime-switching Gaussian copula. The estimated parameters of this model are shown in Table 5.

Table 4: P-values (in percentage) of the different regime-switching copula families, together with the computation time in seconds.

	Copula family									
	Clayton		Frank		Gaussian		Gumbel		Student	
	Number of regimes		Number of regimes		Number of regimes		Number of regimes		Number of regimes	
	1	2	1	2	1	2	1	2	1	2
P-value	0.0	1.0	0.6	0.1	0.0	**9.4**	0.0	0.0	4.8	**12.8**
Sec.	195	1272	128	490	175	712	198	1342	1715	15386

Table 5: Estimated parameters for the regime-switching Gaussian copula model with two regimes. Here, τ is Kendall's tau, $\rho = \sin(\pi\tau/2)$ is the correlation coefficient of the copula, and ν is the stationary distribution of the regimes.

Parameter	Regime 1	Regime 2
τ	0.0859	0.5816
ρ	0.1346	0.7917
ν	0.5209	0.4791
P	$\begin{pmatrix} 0.7414 & 0.2586 \\ 0.2812 & 0.7188 \end{pmatrix}$	

5.2 Bivariate option pricing

In order to price an option with payoff $\Phi(S_{1n}, S_{2n})$ over n trading days, we perform a Monte Carlo simulation under a risk neutral measure. First, as in van den Goorbergh et al. (2005), we assume that the selected regime-switching copula model with parameters given in Table 5 is also valid under the risk neutral measure. Next, for the dynamic models of both time series, we assume that we still have Gaussian HMM, but with new parameters, namely $\tilde{\mu}_{jk} = r - \frac{\sigma_{jk}^2}{2}$, $\tilde{\sigma}_{jk} = \sigma_{jk}$, and $\tilde{Q}^{(j)} = Q^{(j)}$, where r is the risk free daily interest rate. This way, under the risk neutral measure, the discounted prices $e^{-rt}S_{jt} = e^{\sum_{i=1}^{t}(X_{ji}-r)}$ form a martingale, for each $j = 1, 2$.

The following steps illustrate the procedure to evaluate a European option with payoff Φ in the case of a general regime-switching copula with ℓ regimes, where each univariate time series is modeled by a Gaussian HMM with m_j regimes and parameters $\tilde{\mu}_{j1}, \ldots, \tilde{\mu}_{j\ell_j}$, $\tilde{\sigma}_{j1}, \ldots, \tilde{\sigma}_{j\ell_j}$, $\tilde{Q}^{(j)}$:

1. Generate $\mathbf{U}_t$, $t \in \{1, \ldots, n\}$, from the regime-switching copula model.
2. For $t \in \{1, \ldots, n\}$, and $j = 1, 2$, compute the conditional distribution function F_{jt} under the risk neutral measure, and set $X_{jt} = F_{jt}^{-1}(U_{jt})$.
3. For $j = 1, 2$, compute $S_{jn} = e^{\sum_{t=1}^{n} X_{ji}}$.
4. Repeat N times steps $1 - 3$, in order to get N independent values of (S_{1n}, S_{2n}).

The value of the option is then approximated by the average of the discounted values $e^{-rn}\Phi(S_{1n}, S_{2n})$.

To evaluate the put-on-max option, we used $N = 10000$ simulations, with a maturity of $n = 20$ trading days and a risk free rate $r = 4\%$. Figure 3 displays the price of the option as a function of the strike K. The price of the option using a 1-regime Gaussian copula is also shown, the Kendall's tau is taken to be 0.32, which is its estimated value. The prices given by the 2-regime Gaussian copula are lower that those given by the 1-regime Gaussian copula. This might be due to the fact that according to Table 5, about 52% of the time, we use a Gaussian copula with a very low Kendall's tau.

6 Conclusion

In this paper, we proposed a methodology to select the number of regimes for a regime-switching copula model based on goodness-of fit tests. This methodology can also be used for the univariate HMM and mixtures of copula models. Through Monte Carlo simulations, we showed that the proposed sequential procedure for selecting the number of regimes works when the sample size is large enough. As an

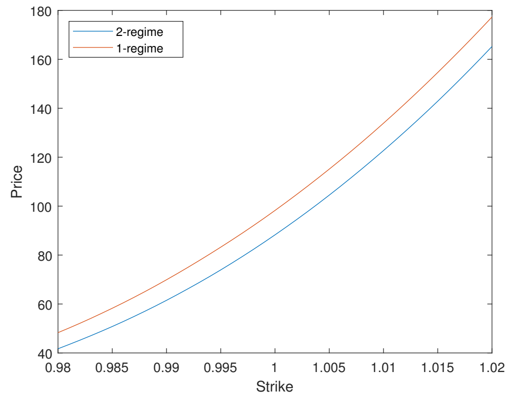

Figure 3: Put-on-max prices for $n = 20$ trading days maturity, as a function of the strike under a (1-regime) Gaussian copula and a 2-regime Gaussian copula.

example of application, we evaluated a European put-on-max option, but the proposed methodology can also be applied to a wide range of multivariate options. The empirical results emphasize the importance of choosing the appropriate number of regimes for the regime-switching copula model.

Appendices

Appendix A Estimation for general regime-switching models

A.1 E-Step

Set $\tilde{\boldsymbol{\theta}} = (\tilde{\boldsymbol{\beta}}_1, \ldots, \tilde{\boldsymbol{\beta}}_\ell, \tilde{P})$. Then, according to (Rémillard, 2013, Appendix 10.A),

$$\begin{aligned} Q_y(\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta}) &= \mathbb{E}_{\boldsymbol{\theta}}\{\log f_{\tilde{\boldsymbol{\theta}}}(\tau, Y)|Y = y\} \\ &= \sum_{t=1}^{n} \sum_{j\in S} \sum_{k\in S} \mathbb{P}_{\boldsymbol{\theta}}(\tau_{t-1} = j, \tau_t = k|Y = y) \log \tilde{P}_{jk} + \sum_{t=1}^{n} \sum_{j\in S} \mathbb{P}_{\boldsymbol{\theta}}(\tau_t = j|Y = y) \log g_{\tilde{\boldsymbol{\beta}}_j}(y_t) \\ &= \sum_{t=1}^{n} \sum_{j\in S} \sum_{k\in S} \Lambda_{\boldsymbol{\theta},t}(j,k) \log \tilde{P}_{jk} + \sum_{t=1}^{n} \sum_{j\in S} \lambda_{\boldsymbol{\theta},t}(j) \log g_{\tilde{\boldsymbol{\beta}}_j}(y_t), \end{aligned}$$

where $\lambda_{\boldsymbol{\theta},t}(j) = P(\tau_t = j|Y = y)$ and $\Lambda_{\boldsymbol{\theta},t}(j,k) = P(\tau_{t-1} = j, \tau_t = k|Y = y)$, for all $t \in \{1, \ldots, n\}$ and $j, k \in S$. Next, define for all $j \in S$, $\bar{\eta}_{\boldsymbol{\theta},n}(j) = 1/\ell$, $\eta_{\boldsymbol{\theta},0}(j) = 1/\ell$,

$$\begin{aligned} \bar{\eta}_{\boldsymbol{\theta},t}(j) &= Prob(\tau_t = j|y_{t+1}, \ldots, y_n), \quad t = 1, \ldots, n-1, \\ \eta_{\boldsymbol{\theta},t}(j) &= Prob(\tau_t = j|y_1, \ldots, y_t), \quad t = 1, \ldots, n. \end{aligned}$$

It follows easily that for $t = 1, \ldots, n$, $\eta_t(j) = \frac{g_{\boldsymbol{\beta}_j}(y_t)}{Z_{t|t-1}} \sum_{i=1}^{\ell} \eta_{t-1}(i) P_{ij}$, where

$$Z_{t|t-1} = \sum_{j=1}^{\ell} g_{\boldsymbol{\beta}_j}(y_t) \sum_{i=1}^{\ell} \eta_{t-1}(i) P_{ij}.$$

Next, for all $i \in \{1, \ldots, l\}$, and for all $t = 0, \ldots, n-1$,

$$\bar{\eta}_{\boldsymbol{\theta},t}(i) = \frac{\sum_{\beta=1}^{\ell} \bar{\eta}_{\boldsymbol{\theta},t+1}(\beta) P_{i\beta} g_{\boldsymbol{\beta}_\beta}(y_{t+1})}{\sum_{k=1}^{\ell} \sum_{\beta=1}^{\ell} \bar{\eta}_{\boldsymbol{\theta},t+1}(\beta) P_{k\beta} g_{\boldsymbol{\beta}_\beta}(y_{t+1})},$$

$$\lambda_{\boldsymbol{\theta},t}(i) = \frac{\eta_{\boldsymbol{\theta},t}(i)\bar{\eta}_{\boldsymbol{\theta},t}(i)}{\sum_{k=1}^{l}\eta_{\boldsymbol{\theta},t}(k)\bar{\eta}_{\boldsymbol{\theta},t}(k)}.$$

Hence, for all $i,j \in \{1,\ldots,l\}$, and for all $t = 1,\ldots,n$,

$$\Lambda_{\boldsymbol{\theta},t}(i,j) = \frac{P_{ij}\eta_{\boldsymbol{\theta},t-1}(i)\bar{\eta}_{\boldsymbol{\theta},t}(j)g_{\boldsymbol{\beta}_j}(y_t)}{\sum_{k=1}^{l}\sum_{\beta=1}^{l}P_{k\beta}\eta_{\boldsymbol{\theta},t-1}(k)\bar{\eta}_{\boldsymbol{\theta},t}(\beta)g_{\boldsymbol{\beta}_\beta}(y_t)}.$$

As a result, for all $i \in \{1,\ldots,l\}$, and for every $t = 1,\ldots,n$, $\sum_{j=1}^{l}\Lambda_{\boldsymbol{\theta},t}(i,j) = \lambda_{\boldsymbol{\theta},t-1}(i)$.

A.2 M-Step

For this step, given $\boldsymbol{\theta}^{(k)}$, $\boldsymbol{\theta}^{(k+1)}$ is defined as $\boldsymbol{\theta}^{(k+1)} = \arg\max_{\boldsymbol{\theta}} Q_y\left(\boldsymbol{\theta},\boldsymbol{\theta}^{(k)}\right)$. Setting $\lambda_t^{(k)}(i) = \lambda_{\boldsymbol{\theta}^{(k)},t}(i)$ and $\Lambda_t^{(k)}(i,j) = \Lambda_{\boldsymbol{\theta}^{(k)},t}(i,j)$, it follows from Section A.1 that

$$\boldsymbol{\theta}^{(k+1)} = \arg\max_{\boldsymbol{\theta}} \sum_{t=1}^{n}\sum_{i,j\in S}\Lambda_t^{(k)}(i,j)\log P_{ij} + \sum_{t=1}^{n}\sum_{i\in S}\lambda_t^{(k)}(i)\log g_{\boldsymbol{\beta}_i}(y_t).$$

Using Lagrange multipliers, the function to maximize is $h(\boldsymbol{\theta},\psi)$, where $\psi = (\psi_1,\ldots,\psi_\ell)$, and

$$h(\boldsymbol{\theta},\psi) = \sum_{t=1}^{n}\sum_{i,j\in S}\Lambda_t^{(k)}(i,j)\log P_{ij} + \sum_{t=1}^{n}\sum_{i\in S}\lambda_t^{(k)}(i)\log g_{\boldsymbol{\beta}_i}(y_t) + \sum_{i=1}^{l}\psi_i\left(1-\sum_{j=1}^{\ell}P_{ij}\right).$$

For $i,j \in S$ we have $\frac{\partial h}{\partial P_{i,j}} = \sum_{t=1}^{n}\Lambda_t^{(k)}(i,j)\frac{1}{P_{ij}} - \psi_i$. As a result, for any $i,j \in S$, the partial derivative of h with respect to P_{ij} is zero if and only if $\psi_i P_{ij} = \sum_{t=1}^{n}\Lambda_t^{(k)}(i,j)$. Summing over j yields that

$$\psi_i = \sum_{j=1}^{\ell}\psi_i P_{ij} = \sum_{j=1}^{\ell}\sum_{t=1}^{n}\Lambda_t^{(k)}(i,j) = \sum_{t=1}^{n}\lambda_{t-1}^{(k)}(i) = \sum_{t=1}^{n}\lambda_{\boldsymbol{\theta}^{(k)},t-1}(i).$$

Hence $P_{ij}^{(k+1)} = \sum_{t=1}^{n}\Lambda_t^{(k)}(i,j)\Big/\sum_{t=1}^{n}\lambda_{t-1}^{(k)}(i)$. Also, maximizing h with respect to $\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_\ell$ amounts to maximize $\sum_{t=1}^{n}\sum_{i=1}^{\ell}\lambda_t^{(k)}(i)\log g_{\boldsymbol{\beta}_i}(y_t)$ with respect to $\boldsymbol{\beta}_i$, for all $i \in S$.

Appendix B Estimation for general mixture models

This model is a particular case of regime-switching where $P_{ij} = \nu_j$, $j \in \{1,\ldots,\ell\}$. So, under this model, τ_t is a sequence of iid observations with distribution $\nu = (\nu_1,\ldots,\nu_\ell)$. The algorithm described previously can then be simplified. To this end, set $\boldsymbol{\theta} = (\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_\ell,\nu)$. The joint density of $\tau = (\tau_1,\ldots,\tau_n)$ and Y is $f_{\boldsymbol{\theta}}(\tau,Y) = \left(\prod_{t=1}^{n}\nu_{\tau_t}\right)\times\prod_{t=1}^{n}g_{\boldsymbol{\beta}_{\tau_t}}(Y_t)$, yielding

$$\log f_{\boldsymbol{\theta}}(\tau,Y) = \sum_{t=1}^{n}\log P_{\tau_{t-1},\tau_t} + \sum_{t=1}^{n}\log g_{\boldsymbol{\beta}_{\tau_t}}(Y_t).$$

B.1 E-Step

Set $\tilde{\boldsymbol{\theta}} = (\tilde{\boldsymbol{\beta}}_1,\ldots,\tilde{\boldsymbol{\beta}}_\ell,\tilde{\nu})$. Then, according to the previous computations,

$$Q_y(\tilde{\boldsymbol{\theta}},\boldsymbol{\theta}) = \mathbb{E}_{\boldsymbol{\theta}}\{\log f_{\tilde{\boldsymbol{\theta}}}(\tau,Y)|Y=y\} = \sum_{t=1}^{n}\sum_{j\in S}\lambda_{\boldsymbol{\theta},t}(j)\left(\log\tilde{\nu}_j + \log g_{\tilde{\boldsymbol{\beta}}_j}(Y_t)\right), \tag{9}$$

where $\lambda_{\boldsymbol{\theta},t}(j) = P_{\boldsymbol{\theta}}(\tau_t = j|Y=y) = \frac{\tilde{\nu_j}g_{\tilde{\boldsymbol{\beta}_{\tau_t}}}(y_t)}{\sum_{k=1}^{\ell}\tilde{\nu_k}g_{\tilde{\boldsymbol{\beta}_k}}(y_t)}$ for all $t \in \{1,\ldots,n\}$ and $j \in S$.

B.2 M-Step

For this step, given $\boldsymbol{\theta}^{(k)}$, $\boldsymbol{\theta}^{(k+1)}$ is defined as $\boldsymbol{\theta}^{(k+1)} = \arg\max_{\boldsymbol{\theta}} Q_y\left(\boldsymbol{\theta}, \boldsymbol{\theta}^{(k)}\right)$. Setting $\lambda_t^{(k)}(i) = \lambda_{\boldsymbol{\theta}^{(k)},t}(i)$, one obtains

$$
\begin{aligned}
Q_y(\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta}) &= \sum_{t=1}^{n}\sum_{j=1}^{\ell} \lambda_t^{(k)}(j) \left(\log \tilde{\nu}_j + \log g_{\tilde{\boldsymbol{\beta}}_j}(Y_t)\right) \\
&= \sum_{t=1}^{n}\sum_{j=1}^{\ell} \lambda_t^{(k)}(j) \log \tilde{\nu}_j + \sum_{t=1}^{n}\sum_{j=1}^{\ell} \lambda_t^{(k)}(j) \log g_{\tilde{\boldsymbol{\beta}}_j}(Y_t).
\end{aligned}
$$

For $j \in S$ we have, $\frac{\partial Q_y}{\partial \tilde{\nu}_j} = \frac{n}{\tilde{\nu}_j} \sum_{t=1}^{n} \lambda_t^{(k)}(j)$. Hence $\tilde{\nu_j}^{(k+1)} = \frac{\sum_{t=1}^{n} \lambda_t^{(k)}(j)}{n}$, $j \in \{1, \ldots, l\}$, and

$$
\tilde{\boldsymbol{\beta}_j}^{(k+1)} = \arg\max_{\tilde{\boldsymbol{\beta}}_j} \sum_{t=1}^{n}\sum_{j=1}^{\ell} \lambda_t^{(k)}(j) \log g_{\tilde{\boldsymbol{\beta}}_j}(y_t).
$$

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