Les Cahiers du GERAD ISSN: 0711–2440

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J. Brimberg
A. Schöbel

G–2018–03

January 2018

La publication de ces rapports de recherche est rendue possible grâce au soutien de HEC Montréal, Polytechnique Montréal, Université McGill, Université du Québec à Montréal, ainsi que du Fonds de recherche du Québec – Nature et technologies.

The publication of these research reports is made possible thanks to the support of HEC Montréal, Polytechnique Montréal, McGill University, Université du Québec à Montréal, as well as the Fonds de recherche du Québec – Nature et technologies.

Dépôt légal – Bibliothèque et Archives nationales du Québec, 2018
– Bibliothèque et Archives Canada, 2018

Legal deposit – Bibliothèque et Archives nationales du Québec, 2018
– Library and Archives Canada, 2018

GERAD HEC Montréal
3000, chemin de la Côte-Sainte-Catherine
Montréal (Québec) Canada H3T 2A7

Tél. : 514 340-6053
Téléc. : 514 340-5665
info@gerad.ca
www.gerad.ca

When closest is not always the best: The distributed p-median problem

Jack Brimberg [a]

Anita Schöbel [b]

[a] *GERAD & Department of Mathematics and Computer Science, Royal Military College of Canada*

[b] *Faculty for Mathematics and Computer Science, Georg-August-Universität Göttingen*

jack.brimberg@rmc.ca
schoebel@math.uni-goettingen.de

January 2018
Les Cahiers du GERAD
G–2018–03

Abstract: The classical p-median problem assumes that service to customers is always provided by the closest facility, while in practice, customers often interact for a variety of reasons with several of the facilities (not just the closest). In this paper we introduce the concept of a distribution rule for modeling such flows, and use it to formulate a new class of median problems which we call the "distributed" p-median problem. Different types of distribution rule are investigated leading to some interesting properties. For example, if the weights are increasing (i.e., assigned flows are greater to facilities that are further away) the problem can be solved in polynomial time as a 1-median problem. For decreasing weights, of which the classical p-median is a special case, we obtain generalizations of the standard continuous and discrete models, which in turn lead to a broader interpretation of median points. Some small numerical examples are given to illustrate the concepts.

1 Introduction

The classical p-median problem whether in discrete or continuous space requires finding the locations of p new facilities in order to minimize a weighted sum of distances from a given set of customers, represented as vertices of a graph or fixed points in continuous space, to their 'closest' facilities. Thus, the basic assumption is made that each customer will be served by a single facility — that one that is 'closest' to it. The p-median problem may be considered the most important and widely studied model in facility location theory. See, for example, the relevant chapters in [17, 6, 14, 8, 10, 20, 22, 16] as an introduction into this topic, as well as numerous surveys including [13, 23, 24, 25, 2] and references therein. The earliest attempts to solve the problem include the well-studied vertex interchange heuristic by Teitz and Bart [26] for the discrete model and alternating locate-allocate heuristic by Cooper [4, 5] for the continuous version; also see seminal papers by Miehle [19] and Hakimi [11, 12] and an interesting history of the continuous single-facility problem in Wesolowsky [28]. For a synopsis of more recent solution approaches, see, e.g., [3, 21, 2, 9].

Extensions (or generalizations) include the capacitated version of the p-median problem. In this case, flow may not be to the nearest facility because of capacity restrictions, but the principle of using the 'closest available' facility still applies, e.g., see [1]. A more recent extension, referred to as the *ordered p-median problem*, still assumes customer flows are routed entirely to the closest facility. However, these closest distances are ordered from smallest to largest and weights $(\lambda_1, \ldots, \lambda_n)$ are applied to them, where n denotes the number of customers. Under this framework, the p-median problem is a special case with $\lambda = (1, \ldots, 1)$ and the p-center problem another special case with $\lambda = (0, \ldots, 0, 1)$. We refer to [22] for an extensive treatment of ordered median problems.

This paper presents an entirely new version of the p-median problem where the flow (or demand) of each customer is divided and routed to as many as all p facilities. The distribution rule is specified by a vector $\lambda = (\lambda_1, \ldots, \lambda_p)$ where parameter λ_k is applied to the k^{th}-closest facility to customer j, $k \in 1, \ldots, p$, $j \in 1, \ldots, n$. Thus, the classical p-median problem in which each customer visits its closest facility becomes a special case with $\lambda = (1, 0, \ldots, 0)$. We believe this new interpretation of the p-median problem has some useful applications, for example:

1. A customer does not always use its closest facility. There may be a number of reasons for this. The customer may like to have some variety of choice even when facilities are assumed to be homogeneous. Thus, the customer may wander some times to his/her second closest or third closest facility (shopping mall, store, ...). The customer may also do this for social reasons, such as meeting friends. A distribution rule other than the 'closest' facility rule may capture customer preferences more realistically. This is, e.g., also done in public transportation where so-called *logit models* [7] are used to reflect the route choice preferences of passengers.
2. The 'closest' facility may not always be available. For example, it may be subject to breakdowns or regular closures for preventive maintenance. The customer flow will be diverted to the next closest facility available in this case. The distribution rule proposed here is a first way of approximating such a scenario.
3. The new model we present gives another interpretation, or definition, of the *median point*. By examining different distribution rules, the decision maker will have a number of different median-type solutions to choose from, instead of just one. This may lead to a better decision based on other considerations such as robustness, redundancy, and so on, and/or more equitable solutions than the classical model.

The rest of the paper is organized as follows. Section 2 presents the model formulation. Section 3 identifies some special cases, including two distribution rules that reduce the problem to finding a single median point. Section 4.2 presents an important sub-class of the problem where the weights assigned to the ordered distances are non-increasing ($\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$). This leads to improved mathematical formulations for both discrete and continuous versions of the problem and to some interesting properties. A few numerical illustrations are provided in Section 5. Section 6 provides conclusions and ideas for further research.

2 The distributed p-median problem

Let us denote $[n] = \{1, \ldots, n\}$ for any natural number n.

We first consider the classical continuous or discrete p-median problem in the space $\mathfrak{X}$: We are given a set of fixed points (demand points, customers) denoted by $\mathcal{A} = \{A_1, \ldots, A_n\} \subseteq \mathfrak{X}$ with associated weights $w_j \geq 0$, $j \in [n]$, representing customer demands (or flows). We also have a distance function $d(\cdot, \cdot)$ which is used to calculate the distance between any two points in $\mathfrak{X}$. We are looking for a set of p new locations $X = \{X_1, \ldots, X_p\} \subseteq \mathfrak{X}$, where $X_i \in \mathfrak{X}$ is the unknown location of facility i, $i \in [p]$. The (classical) p-median problem p-Med may now be written as

$$(p\text{-med}) \qquad \min_{\{X_1,\ldots,X_p\}\subseteq\mathfrak{X}} f(X) = \sum_{j=1}^{n} w_j \min_{i\in[p]} d(X_i, A_j). \tag{1}$$

A special case is the 1-median problem in which we look for only $p = 1$ facility X_1, i.e.,

$$(1\text{-med}) \qquad \min_{X_1\in\mathfrak{X}} f(X_1) = \sum_{j=1}^{n} w_j d(X_1, A_j). \tag{2}$$

An optimal solution to (1-med) is denoted as $X^*_{1\text{-med}}$.

The p-median problem is defined in the continuous and in the discrete case.

***p*-med-c** In the continuous case, it is often also referred to as the *multi-source Weber problem* (or *continuous location-allocation problem*). The demand points and the new facilities are all in the space $\mathfrak{X} = \mathbb{R}^N$. Typically, the distance function is given by the Euclidean or rectangular norm, or more generally by any norm $\|\cdot\|$, or even a gauge, and the location problem occurs in the plane ($N = 2$). The objective is to find p new locations in $\mathbb{R}^N$ in order to minimize the weighted sum of distances between each customer and its closest facility.

***p*-med-d** The discrete case is usually modeled as a connected network $G = (V, E)$. The space $\mathfrak{X} = V$ consists of n nodes representing the n customers' locations $A_1, \ldots, A_n$. The set of edges E represents, e.g., roads. Each edge $\{i, j\} \in E$ has an associated shortest path distance d_{ij} obtained directly from the network. The objective in the classical discrete p-median problem is to select p of these nodes to open facilities $X_1, \ldots, X_p$ at in order to minimize a weighted sum of 'shortest path' distances between each customer and its closest facility.

Now let us consider the new generalized model where customer demand (or flow) is distributed by some specified rule among the p facilities instead of being served uniquely by the closest facility as shown in (1). For the sake of brevity we still consider $\mathfrak{X}$ here as the continuous space $\mathbb{R}^N$ or as the set of nodes of a network in the discrete case. This distribution rule, which applies to each customer, is denoted by vector

$$\begin{aligned} \lambda &= (\lambda_1, \ldots, \lambda_p), && (3) \\ \text{where} \quad \sum_{k=1}^{p} \lambda_k &= 1, \ \lambda_k \geq 0, k \in [p], && (4) \end{aligned}$$

and λ_k represents the proportion of demand w_j assigned to the k^{th}-closest facility to customer j, $j \in [n]$. For the construction of the new model, we define

$$X_{(j,k)} \qquad \text{as the facility that is the } k^{th}\text{-closest one to fixed location } A_j, j \in [n]; k \in [p], \tag{5}$$

where ties are broken arbitrarily. Formally, $(X_{(j,1)}, X_{(j,2)}, \ldots, X_{(j,p)})$ is a permutation of the new facilities $(X_1, X_2 \ldots, X_p)$ such that

$$d(X_{(j,1)}, A_j) \leq d(X_{(j,2)}, A_j) \leq \ldots \leq d(X_{(j,p)}, A_j)$$

holds. In particular, we have

$$\{X_{(j,1)}, X_{(j,2)}, \dots, X_{(j,p)}\} = \{X_1, X_2, \dots, X_p\}. \tag{6}$$

Note that $X_{(j,k)}$ is a function of the set of *all* location decisions $X = \{X_1, \dots, X_p\}$, i.e., in a precise way we would have to write $X_{(j,k)}(X)$, but we neglect this dependence if it is clear which set is meant. We denote

$$d(X_{(j,k)}) := d(X_{(j,k)}, A_j) \tag{7}$$

as the distance from A_j to its k^{th}-closest facility. Per definition we then have

$$d(X_{(j,1)}) \leq d(X_{(j,2)}) \leq \ldots \leq d(X_{(j,p)}),$$

in particular

$$d(X_{(j,1)}) \leq d(X_i, A_j) \leq d(X_{(j,p)}) \text{ for all } i \in [p], j \in [n]. \tag{8}$$

Given a distribution rule λ (as in (3) and (4)), the new model which we term the *distributed p-median problem*, takes the form:

$$\text{(D-}p\text{-med)} \qquad \min_{\{X_1,\dots,X_p\}\subseteq \mathfrak{X}} f_D(X;\lambda) = \sum_{j=1}^{n}\sum_{k=1}^{p} \lambda_k w_j d(X_{(j,k)}) \tag{9}$$

where $X_{(j,k)}$ are defined in (5) and $d(X_{(j,k)}) = d(X_{(j,k)}, A_j)$ has been defined in (7). Its objective function $f_D(X;\lambda)$ depends not only on X, but also on (the given) distribution rule λ. When it is clear which λ is meant, we will just write $f_D(X)$.

Note that if $\lambda = (1, 0, \dots, 0)$, then

$$f_D(X) = \sum_{j=1}^{n} w_j d(X_{(j,1)}) = f(X), \tag{10}$$

so that p-med is now interpreted as a special case of D-p-med with its own special distribution rule λ.

Expressing D-p-med as a mathematical program is not straightforward because of the ordering of distances of each customer to the facility locations $\{X_1, \dots, X_p\}$. To accomplish this task, we borrow from the insights of the mathematical programming formulation of the classical p-med-c. Still being in an arbitrary space $\mathfrak{X}$ define

$$z_{ij}^k = \begin{cases} 1, & \text{if } X_i \text{ is the } k^{th}\text{-closest facility to } A_j \\ 0, & \text{otherwise,} \end{cases}$$

for $i \in [p]$, $j \in [n]$, $k \in [p]$. We then can formulate D-p-med as follows.

$$\begin{aligned}
\min\ f_D(X) &= \sum_{k=1}^{p}\sum_{j=1}^{n}\sum_{i=1}^{p} \lambda_k w_j z_{ij}^k d(X_i, A_j) && (11)\\
\text{s.t.} \quad \sum_{k=1}^{p} z_{ij}^k &= 1,\ i \in [p], j \in [n] && (12)\\
\sum_{i=1}^{p} z_{ij}^k &= 1,\ k \in [p], j \in [n] && (13)\\
\sum_{i=1}^{p} z_{ij}^{k+1} d(X_i, A_j) &\geq \sum_{i=1}^{p} z_{ij}^k d(X_i, A_j),\ j \in [n], k \in [p-1] && (14)\\
z_{ij}^k &\in \{0,1\},\ i \in [p], j \in [n], k \in [p]. && (15)\\
X_i &\in \mathfrak{X},\ i \in [p] && (16)
\end{aligned}$$

Note that constraints (12) and (13) are required in order to establish for each customer j a one-to-one correspondence between ordered distances and facilities. The constraints in (14) establish the correct ordered sequence of distances.

2.1 A nonlinear mixed binary program for the continuous case

We now turn to the continuous case $\mathfrak{X} = \mathbb{R}^N$. Note that in this case we can delete the restriction (16) and receive a nonlinear mixed binary program (11)–(15) which could be directly used by a solver. The resulting distributed (continuous) p-median problem is denoted as D-p-med-c and referred to as *distributed multi-source Weber problem* or as *continuous distributed p-median problem*.

Returning to the classical p-med-c with $\lambda = (1, 0, \ldots, 0)$, removing redundant constraints, and substituting $z_{ij} = z_{ij}^1$ for all $i \in [p]$, $j \in [n]$, the formulation of D-p-med-c reduces to:

$$\min\ f(X) = \sum_{j=1}^{n} \sum_{i=1}^{p} w_j z_{ij} d(X_i, A_j) \tag{17}$$

$$\text{s.t.} \quad \sum_{i=1}^{p} z_{ij} = 1,\ j \in [n] \tag{18}$$

$$z_{ij} \in \{0, 1\},\ i \in [p],\ j \in [n]. \tag{19}$$

Furthermore, the minimization of the objective function ensures that each customer will be assigned to its closest facility, and we can relax (19) to

$$z_{ij} \geq 0,\ i \in [p],\ j \in [n]. \tag{20}$$

Substituting $w_{ij} = w_j z_{ij}$ for all $i \in [p]$, $j \in [n]$, we obtain the standard formulation of p-med-c (see, e.g., [17]):

$$\min\ f(X) = \sum_{j=1}^{n} \sum_{i=1}^{p} w_{ij} d(X_i, A_j) \tag{21}$$

$$\text{s.t.} \quad \sum_{i=1}^{p} w_{ij} = w_j,\ j \in [n] \tag{22}$$

$$w_{ij} \geq 0,\ i \in [p],\ j \in [n]. \tag{23}$$

2.2 An integer linear program for the discrete case

We now turn to the discrete model in which we have a network $G = (V, E)$. The space $\mathfrak{X} = V$ consists of the n nodes of the network. The distributed p-median problem for the discrete case $\mathfrak{X} = V$ is called *discrete distributed p-median problem* and denoted as D-p-med-d.

In order to select p of the n nodes we need decision variables

$$x_i = \text{number of facilities to be opened at node } i,$$

and

$$z_{ij}^k = \begin{cases} 1 & \text{if a facility at node } i \text{ is the } k^{th} \text{ closest facility to node } j \\ 0 & \text{otherwise,} \end{cases}$$

$i, j \in [n], k \in [p]$. Note that the z_{ij}^k variables serve an analogous purpose as in the continuous model D-p-med-c. Also note that the x_i variables must be specified as non-negative integers instead of binary in order to allow more than one facility to coincide with the same node, i.e., customers may have, e.g., their closest and their second-closest facilities at the same location. We now present the formulation for D-p-med-d.

$$\min\ f_D(X) = \sum_{k=1}^{p} \sum_{j=1}^{n} \sum_{i=1}^{n} \lambda_k w_j d_{ij} z_{ij}^k \tag{24}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} x_i = p, \tag{25}$$

$$\sum_{k=1}^{p} z_{ij}^k = x_i,\ i,j \in [n], \tag{26}$$

$$\sum_{i=1}^{n} z_{ij}^k = 1,\ k \in [p], j \in [n] \tag{27}$$

$$\sum_{i=1}^{n} d_{ij} z_{ij}^{k+1} \geq \sum_{i=1}^{n} d_{ij} z_{ij}^k,\ j \in [n], k \in [p-1] \tag{28}$$

$$x_i \in \mathbb{N}_0, z_{ij}^k \in \{0,1\},\ i,j \in [n], k \in [p]. \tag{29}$$

Constraint (25) ensures that exactly p facilities are opened. Constraints (26), (27), (28) serve the same purpose as (12), (13), (14) in D-p-med-c. Integer and binary constraints are imposed on the decision variables in (29).

As for the continuous model we would like to see what happens to the formulation when $\lambda = (1, 0, \ldots, 0)$, i.e., the distribution rule for the classical p-median problem (p-med-d for short) applies. Substituting $z_{ij} = z_{ij}^1$ for all $i, j \in [n]$ and eliminating redundant constraints, the problem formulation for p-med-d reduces to the standard form given, e.g., in [14].

$$\min \sum_{j=1}^{n} \sum_{i=1}^{n} w_j d_{ij} z_{ij} \tag{30}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} x_i = p \tag{31}$$

$$z_{ij} \leq x_i,\ i,j \in [n] \tag{32}$$

$$\sum_{i=1}^{n} z_{ij} = 1,\ j \in [n] \tag{33}$$

$$x_i, z_{ij} \in \{0,1\},\ i,j \in [n]. \tag{34}$$

Note that binary constraints may replace integer constraints on the x_i variables here under the standard assumption $p \leq n$. Analogous to p-med-c, we may also replace the binary constraints on the z_{ij} by non-negativity constraints as in (20).

3 Polynomially solvable cases

We already observed that the classical closest-facility rule $\lambda = (1, 0, \ldots, 0)$ turns out to be p-med-c and p-med-d in the continuous and discrete cases, respectively. These two problems are known to be NP-hard, see [18, 15]. In this section two special cases are presented that can be reduced to 1-median problems and hence be solved exactly in polynomial time.

3.1 The uniform distribution rule $\lambda = (\frac{1}{p}, \ldots, \frac{1}{p})$

For the case of uniform weights $\lambda = (\frac{1}{p}, \ldots, \frac{1}{p})$, the objective function (9) of D-p-med can be rewritten as

$$\begin{aligned} f_D(X_1, \ldots, X_p) &= \sum_{k=1}^{p} \sum_{j=1}^{n} \tfrac{1}{p} w_j d(X_{(j,k)}) \\ &= \tfrac{1}{p} \sum_{j=1}^{n} \sum_{k=1}^{p} w_j d(X_{(j,k)}, A_j) \\ &= \tfrac{1}{p} \sum_{j=1}^{n} \sum_{i=1}^{p} w_j d(X_i, A_j), \quad \text{see (6).} \end{aligned}$$

Thus, D-p-med reduces to p independent single-facility location problems. The following result is obvious.

Lemma 1 *An optimal solution of D-p-med with distribution rule $\lambda = (\frac{1}{p}, \ldots, \frac{1}{p})$ places all p facilities at an optimal solution $X^*_{1\text{-}med}$ of the 1-median problem (2).*

3.2 The furthest-facility rule $\lambda = (0, \ldots, 0, 1)$

Each customer is now served entirely by its furthest facility. The objective function (9) of D-p-med reduces to

$$\begin{aligned} f_D(X_1, \ldots, X_p) &= \sum_{j=1}^{n} \sum_{k=1}^{p} \lambda_k w_j d(X_{(j,k)}) \\ &= \sum_{j=1}^{n} \underbrace{\lambda_p}_{=1} w_j d(X_{(j,p)}, A_j) \\ &= \sum_{j=1}^{n} w_j d(X_{(j,p)}, A_j) \\ &\geq \sum_{j=1}^{n} w_j d(X_1, A_j), \text{ see (8)} \\ &\geq \sum_{j=1}^{n} w_j d(X^*_{1\text{-med}}, A_j), \end{aligned}$$

where $X^*_{1\text{-med}}$ as before denotes an optimal solution of the 1-median problem (1-med) of (2).

On the other hand, if all p facilities coincide with $X^*_{1\text{-med}}$, i.e., $X_1 = X_2 = \ldots = X_p = X^*_{1\text{-med}}$ then $X_{(j,p)} = X^*_{1\text{-med}}$ for all $j \in [n]$, and

$$f_D(X_1, \ldots, X_p) = \sum_{j=1}^{n} w_j d(X^*_{1\text{-med}}, A_j).$$

This solves the problem with $\lambda = (0, \ldots, 0, 1)$:

Lemma 2 *An optimal solution of D-p-med with distribution rule $\lambda = (0, \ldots, 0, 1)$ places all p facilities at an optimal solution $X^*_{1\text{-}med}$ of the 1-median problem (2).*

4 The case of monotone weights

We now generalize the special cases which we have considered in the previous section. In particular, we investigate the general cases of increasing and decreasing weights $\lambda_1, \ldots, \lambda_p$.

4.1 Increasing distribution rule

Consider a distribution rule λ where the weights λ_i are increasing, that is,

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_p. \tag{35}$$

We call such a λ an *increasing distribution rule* and denote it by λ_{inc}. The two cases $\lambda = (\frac{1}{p}, \ldots, \frac{1}{p})$ and $\lambda = (0, 0, \ldots, 1)$ considered in Sections 3.1 and 3.2 are both examples of this more general rule; they may in fact be considered extreme cases of increasing distribution rules. For convenience we label the rule $\lambda = (\frac{1}{p}, \ldots, \frac{1}{p})$ as λ_{base}. We now show that the same optimal solution obtained for λ_{base} (and also for $\lambda = (0, \ldots, 0, 1)$), i.e., to locate all p facilities at the 1-median point (see Lemmas 1 and 2), applies for all increasing distribution rules λ_{inc}.

In the following we denote by X an arbitrary solution of D-p-med.

Lemma 3 *Let X denote an arbitrary solution of D-p-med, and let $\lambda_{inc} \neq \lambda_{base}$ be given with $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_p$. Then*

$$f_D(X; \lambda_{inc}) \geq f_D(X; \lambda_{base}).$$

Proof. Since $\lambda_{inc} \neq \lambda_{base}$ there must be a $t \in \{2, \ldots, p\}$ such that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{t-1} \leq \frac{1}{p}$, and $\frac{1}{p} < \lambda_t \leq \lambda_{t+1} \leq \ldots \leq \lambda_p$. Consider a transfer of weights in which we reduce the weight of λ_t to $\frac{1}{p}$ and increase the weights of the lower-indexed components, $\lambda_1, \ldots, \lambda_{t-1}$ accordingly. This is represented by the new (again increasing) distribution rule:

$$\lambda' = (\lambda_1 + \Delta_1, \lambda_2 + \Delta_2, \ldots, \lambda_{t-1} + \Delta_{t-1}, \tfrac{1}{p}, \lambda_{t+1}, \ldots, \lambda_p),$$

where $\Delta_r \geq 0$ for all $r \in [t-1]$, $\sum_{r=1}^{t-1} \Delta_r = \lambda_t - \frac{1}{p}$, and

$$\lambda_1 + \Delta_1 \leq \lambda_2 + \Delta_2 \leq \ldots \leq \lambda_{t-1} + \Delta_{t-1} \leq \tfrac{1}{p}.$$

This transfer of weights to lower-indexed components of λ cannot deteriorate the objective value of solution X, since larger weights are now applied to closer facilities of each customer point, the weight applied to the t^{th}-closest facility decreases, and the remaining weights remain the same. That is, $f_D(X; \lambda_{inc}) \geq f_D(X; \lambda')$, and λ' differs in one component less from λ_{base} since $\lambda'_t = \frac{1}{p}$. We repeat this procedure with a finite number ($\leq p-1$) of analogous adjustments. None of these adjustments can change the weight of λ_t further, hence in each iteration one more component gets the weight $\frac{1}{p}$ until we end up with the uniform distribution rule λ_{base}. We conclude $f_D(X; \lambda_{inc}) \geq f_D(X; \lambda') \geq f_D(X; \lambda'') \geq \ldots \geq f_D(X; \lambda_{base})$. □

Lemma 4 *Consider any solution X for D-p-med where all p facilities coincide at the same point, i.e., $X_1 = X_2 = \ldots = X_p$. Let $\lambda^1 \neq \lambda^2$ denote two different arbitrary distribution rules. Then,*

$$f_D(X; \lambda^1) = f_D(X; \lambda^2).$$

Proof. For arbitrary λ we have

$$\begin{aligned} f_D(X;\lambda) &= \sum_{j=1}^{n} \sum_{k=1}^{p} \lambda_k w_j d(X_{(j,k)}) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{p} \lambda_k w_j d(X_1, A_j) \\ &= \sum_{j=1}^{n} w_j d(X_1, A_j) \underbrace{\sum_{k=1}^{p} \lambda_k}_{=1} \\ &= \sum_{j=1}^{n} w_j d(X_1, A_j) \end{aligned}$$

which is independent of λ and hence $f_D(X; \lambda^1) = f_D(X; \lambda^2)$. □

Theorem 5 *Let λ_{inc} be any arbitrary increasing distribution rule. Then an optimal solution X^* of D-p-med with distribution rule λ_{inc} places all p facilities at an optimal solution $X^*_{1\text{-}med}$ of the 1-median problem (2).*

Proof. Let X denote an arbitrary solution to D-p-med and X^*_{base} denote the optimal solution for λ_{base} with $X_1 = X_2 = \cdots = X_p = X^*_{\text{1-med}}$ (see Lemma 1). Then

$$\begin{aligned} f_D(X, \lambda_{inc}) &\geq f_D(X, \lambda_{base}) \text{ see Lemma 3} \\ &\geq f_D(X^*_{base}, \lambda_{base}) \\ &= f_D(X^*_{base}, \lambda_{inc}) \text{ see Lemma 4.} \end{aligned}$$

Therefore, placing all facilities at $X^*_{\text{1-med}}$ is an optimal solution for distribution rule λ_{inc}. □

4.2 Decreasing distribution rule

We now consider an arbitrary distribution rule $\lambda_{dec} = (\lambda_1, \ldots, \lambda_p)$ such that

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p. \tag{36}$$

As before, $\lambda_k \geq 0$ and $\sum_{k=1}^{p} \lambda_k = 1$. The classical closest-facility rule $\lambda = (1, 0, \ldots, 0)$ is a special case of this decreasing distribution rule; so is the uniform rule $\lambda = (\frac{1}{p}, \ldots, \frac{1}{p})$ of Section 3.1. In fact, we may consider the closest-facility rule to be an extreme case, since in many situations noted in the Introduction, not all customer flows will go to their closest facilities. On the other hand, the decreasing rule makes good practical sense from the view-point that distance to service is a factor and, all else being equal, customers tend to prefer facilities that are closer to them, but not necessarily always the closest. This also holds for other contexts, e.g., distribution of passengers to routes in logit models where it is assumed that the majority of passengers chooses the best route, but still a large part will decide to use the second-best route, then the third-shortest route, and so on.

We first propose the following simplification of the formulation (11)–(16). To this end, let z_{ij}^k denote the fraction of assigned flow from customer j to facility i, acting as its k^{th}-closest facility. The formulation for λ_{dec} which we term the *distributed p-median problem with decreasing weights* D↓-p-med is now given by:

$$\min \; f_{D\downarrow}(X) = \sum_{k=1}^{p} \sum_{j=1}^{n} \sum_{i=1}^{p} \lambda_k w_j z_{ij}^k d(X_i, A_j) \tag{37}$$

$$\text{s.t.} \quad \sum_{k=1}^{p} z_{ij}^k = 1, \; i \in [p], j \in [n] \tag{38}$$

$$\sum_{i=1}^{p} z_{ij}^k = 1, \; k \in [p], j \in [n] \tag{39}$$

$$z_{ij}^k \geq 0, \; i \in [p], j \in [n], k \in [p]. \tag{40}$$

$$X_i \in \mathfrak{X}, \; i \in [p] \tag{41}$$

Note that a similar reasoning applies as in the classical p-median problem p-med for identifying the closest facility in order to show that the formulation preserves the correct ordering of distances from a customer to the facilities. In the proof below we also see why it is necessary for the λ_k to be decreasing.

Theorem 6 *The formulation* (37)–(41) *is a correct formulation for D↓-p-med.*

Proof. We have to show that an optimal solution of D↓-p-med exists such that the whole flow $\lambda_k w_j$ is assigned to the k^{th}-closest facility to customer j for all $j \in [n], k \in [p]$.

To this end, consider any solution $X = \{X_1, \ldots, X_p\}$, and suppose that $\alpha := z_{tj'}^1 > 0$ for some customer j' and some facility X_t that is <u>not</u> the (first-)closest facility to j'. Then transfer $\alpha\lambda_1 w_{j'}$ flow from facility X_t to the (first-)closest facility to j', say X_r; i.e.,

$$\begin{aligned} z_{rj'}^1 &\rightarrow z_{rj'}^1 + \alpha, \\ z_{tj'}^1 &\rightarrow z_{tj'}^1 - \alpha = 0. \end{aligned}$$

Without loss of generality, assume that $z_{rj'}^2 \geq \alpha$, otherwise a straightforward adjustment to the flow transfers shown below can be made.

To maintain feasibility (i.e., to ensure that (38), (39), (40) are satisfied) the following additional transfers are made:

$$\begin{aligned} z_{rj'}^2 &\rightarrow z_{rj'}^2 - \alpha, \\ z_{tj'}^2 &\rightarrow z_{tj'}^2 + \alpha. \end{aligned}$$

The change in objective value resulting from the above transfers is

$$\begin{aligned}\Delta f_{D\downarrow} &= \alpha\lambda_1 w_{j'}(d(X_r, A_{j'}) - d(X_t, A_{j'})) + \alpha\lambda_2 w_{j'}(d(X_t, A_{j'}) - d(X_r, A_{j'}))\\ &= \alpha w_{j'}(\lambda_1 - \lambda_2)(d(X_r, A_{j'}) - d(X_t, A_{j'})) \le 0,\end{aligned}$$

since $\lambda_1 \ge \lambda_2$ and $d(X_r, A_{j'}) \le d(X_t, A_{j'})$. Thus, the solution is improved by the above transfers, or at worst, the objective value remains unchanged if $\lambda_1 = \lambda_2$ or $d(X_r, A_{j'}) = d(X_t, A_{j'})$. We proceed in this manner until all the assigned flow to j' $(= \lambda_1 w_{j'})$ goes to its (first-)closest facility X_r, i.e., $z^1_{rj'} = 1$, $z^1_{tj'} = 0$ for all $t \ne r$, and no further adjustments to the $z^1_{ij'}$ variables are required for all $i \in [p]$. Next, we repeat the process for the second-closest facility to j' and so on in sequence all the while generating a series of improving (or equivalent) solutions, until the correct assignment of flows is achieved at j'. Finally, the entire process is repeated individually for each remaining customer j, thereby improving the solution further (or maintaining the status quo), until all flows are correctly assigned to arbitrary solution X. Thus, an optimal solution must exist that preserves the correct order of distances. □

We can directly use Theorem 6 to simplify the mixed binary nonlinear program used to model the continuous problem D-p-med-c for arbitrary λ and receive a nonlinear program with exclusively continuous variables for decreasing weights λ_{dec} given by (37)–(40). The formulation of the discrete distributed p-median problem D-p-med-d (see (24)–(29) in Section 2.2) in the case of decreasing weights λ_{dec} may be simplified in a similar way as in (37)–(41). The new model, referred to as the *distributed discrete p-median problem with decreasing weights* D↓-p-med-d is given below:

$$\min\ f_{D\downarrow}(X) = \sum_{k=1}^{p}\sum_{j=1}^{n}\sum_{i=1}^{n} \lambda_k w_j d_{ij} z^k_{ij} \tag{42}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} x_i = p, \tag{43}$$

$$\sum_{k=1}^{p} z^k_{ij} = x_i,\ \ i,j \in [n], \tag{44}$$

$$\sum_{i=1}^{n} z^k_{ij} = 1,\ \ k \in [p], j \in [n] \tag{45}$$

$$x_i \in \mathbb{N}_0, z^k_{ij} \ge 0,\ \ i,j \in [n], k \in [p]. \tag{46}$$

Theorem 6 is now readily extended to the discrete case given above by (42)–(46), i.e., we receive:

Theorem 7 *The formulation (42)–(46) is a correct formulation for D↓-p-med-d.*

Note that the pure integer linear program (24)–(29) originally formulated for D-p-med-d in Section 2.2 simplifies to a mixed integer linear program with much fewer integer variables (only n instead of $n + n^2p$).

We end this section by stating two consequences which follow from the improved formulations.

Assume $X = \{X_1, \ldots, X_p\}$ is an optimal solution to the distributed p-median problem D↓-p-med either in its discrete or in its continuous version. Then we define p 1-median problems $(1-\mathrm{med}_1), \ldots, (1-\mathrm{med}_p)$, one for each new facility, as follows:

- The demand points of $(1-\mathrm{med}_i)$ are $A_1, \ldots, A_n$ as before.
- The weight w'_j for demand point A_j in problem $(1-\mathrm{med}_i)$ is defined as $w'_j := w_j\lambda_k$ if X_i is the k^{th} closest facility to A_j, i.e., if $X_i = X_{(j,k)}$.

We have the following result.

Lemma 8 *Let $X = \{X_1, \ldots, X_p\}$ be an optimal solution to D↓-p-med. Then X_i is a 1-median for $(1-\mathrm{med}_i)$ for $i = 1, \ldots, p$.*

Proof. Let an optimal solution X be given. Fixing the X_i's in formulation (37)–(41) for D↓-p-med, gives us values for the variables z_{ij}^k, $i, k \in [p], j \in [n]$. From Theorem 6 we know that $z_{ij}^k \in \{0, 1\}$. Fixing these values reduces (37)–(41) to

$$f_{D\downarrow}(X) = \sum_{k=1}^{p}\sum_{j=1}^{n}\sum_{i=1}^{p} \lambda_k w_j z_{ij}^k d(X_i, A_j) = \sum_{i=1}^{p}\sum_{j=1}^{n} w'_{ij} d(X_i, A_j)$$

where w'_{ij} is defined as $w_j \lambda_k$ if $z_{ij}^k = 1$ which (due to (38)) happens exactly once for each pair i, j. $f_{D\downarrow}(X)$ hence decomposes into the sum of p objective functions, each of them describing a problem of type $(1-\text{med}_i)$. We conclude that the location X_i is an optimal solution of $(1-\text{med}_i)$; otherwise relocating it to $X^*_{\text{1-med}}$ would result in an overall improvement of $f_{D\downarrow}(X)$ due to this re-location, and then subsequent re-distribution of flows if required (as in Theorem 6). □

Lemma 8 has two interesting consequences. The first shows the existence of a finite dominating set for D↓-p-med-c when a block norm is used.

Corollary 9 *Assume D↓-p-med-c with a block norm as distance measure. In this case, there always exists an optimal solution $\{X_1, \ldots, X_p\}$ such that each of the new facilities X_i, $i \in [p]$ lies on a vertex of the grid spanned by the fundamental directions through the demand points.*

Proof. It is well known that an optimal solution of the 1-median problem exists which coincides with a grid vertex (see, [27]). The result hence follows from Lemma 8 by choosing a solution in which every X_i, $i \in [p]$ is a median to the associated 1-median problem $(1 - \text{med}_i)$. □

Note that this property need not hold for general distribution rules that are not decreasing as shown in the following example in which we consider the following five demand points in the plane:

$$A_1 = (2, 2), A_2 = (3, 7), A_3 = (8, 2), A_4 = (7, 9), A_5 = (4, 9)$$

We use the Manhattan distance and uniform weights. An optimal solution for D-p-med-c with $p = 3$ new facilities and

$$\lambda = (0.5, 0.1, 0.4)$$

is shown in Figure 1. It consists of the three new facilities

$$X_1 = (3, 7), X_2 = (4, 6), X_3 = (4, 8),$$

from which only X_1 is on a grid point. (Enumerating all grid points shows that this solution is in fact better than any solution which consists of grid points only.)

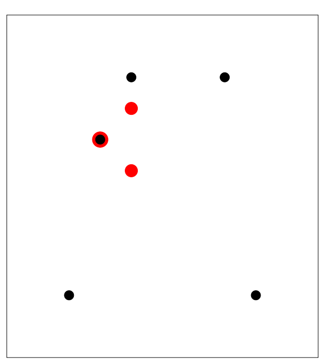

Figure 1: An example for $p = 3$ and the distribution rule $(0.5, 0.1, 0.4)$ in which no optimal solution on grid points exists.

The second consequence allows a local improvement strategy similar to Cooper's algorithm [4] for the classic p-median problem: Start with an arbitrary solution $\{X_1,\ldots,X_p\}$, determine the allocation variables z_{ij}^k, or, equivalently, determine

$$X_{(j,k)} \quad \text{as the facility that is the } k^{th}\text{-closest one to fixed location } A_j, j \in [n]; k \in [p],$$

as already defined in (5), and then solve the p independent problems $(1-\text{med}_i)$, $i \in [p]$ to obtain a new solution $\{X_1',\ldots,X_p'\}$. Due to Lemma 8, we have that $f_D(X';\lambda_{dec}) \leq f_D(X;\lambda_{dec})$. We hence can iterate until no further improvement is found. Of course, this procedure only guarantees a local optimum.

5 Illustration

In this section we illustrate the *distributed p-median problem* $D-\text{p}-med$ using small numerical examples. We selected the Manhattan distance for our computations since it can be linearized easily. The resulting mixed integer linear program was then solved by mosel Xpress.

5.1 A mixed integer linear program for the rectangular distance

To linearize (11)–(15) we first define

$$D_{ij} := d(X_i,A_j) = \|X_i - A_j\|_1$$

which can be formulated linearly for the rectangular distance by requiring

$$\begin{aligned}
X_{i,1} - A_{j,1} + X_{i,2} - A_{j,2} &\leq D_{ij} \text{ for all } i \in [p], j \in [n] & (47)\\
X_{i,1} - A_{j,1} - X_{i,2} + A_{j,2} &\leq D_{ij} \text{ for all } i \in [p], j \in [n] & (48)\\
-X_{i,1} + A_{j,1} + X_{i,2} - A_{j,2} &\leq D_{ij} \text{ for all } i \in [p], j \in [n] & (49)\\
-X_{i,1} + A_{j,1} - X_{i,2} + A_{j,2} &\leq D_{ij} \text{ for all } i \in [p], j \in [n]. & (50)
\end{aligned}$$

Both in the objective function (11) and in constraint (14) we still have the product $z_{ij}^k \cdot D_{ij}$. In order to linearize this we furthermore introduce variables

$$u_{ij}^k := z_{ij}^k \cdot D_{ij}$$

which are set correctly by the constraints

$$\begin{aligned}
u_{ij}^k &\geq D_{ij} + D_{\max}(z_{ij}^k - 1) \text{ for all } i,k \in [p], j \in [n] & (51)\\
u_{ij}^k &\geq 0 \text{ for all } i,k \in [p], j \in [n] & (52)
\end{aligned}$$

if $D_{\max}$ is chosen as the diameter of the set of demand points. The resulting formulation for the case of rectangular distances then is a mixed binary linear program which reads

$$\begin{aligned}
\min\ f_D(X) &= \sum_{k=1}^{p}\sum_{j=1}^{n}\sum_{i=1}^{p} \lambda_k w_j u_{ij}^k & (53)\\
\text{s.t. } \sum_{i=1}^{p} u_{ij}^{k+1} &\geq \sum_{i=1}^{p} u_{ij}^k,\ j \in [n], k \in [p-1] & (54)\\
(12), (13), (47) &- (52), & (55)\\
z_{ij}^k &\in \{0,1\},\ i \in [p], j \in [n], k \in [p] & (56)\\
u_{ij}^k, D_{ij}, X_{i,1}, X_{i,2} &\in \mathbb{R},\ i \in [p], j \in [n], k \in [p]. & (57)
\end{aligned}$$

5.2 A textbook example

We first present a small example with four demand points

$$A_1 = (0,0), A_2 = (10,0), A_3 = (10,5) \text{ and } A_4 = (0,5)$$

only, arranged as vertices of a rectangle with side lengths 5 and 10. The case of $p = 2$ new facilities with a distribution rule $\lambda = (\alpha, 1-\alpha)$ can be solved optimally as follows.

a) for unit weights $w_j = 1$ for $j \in [4]$

$$\begin{cases} X_1^* = (10,0), X_2^* = (10,0) & \text{if } \alpha \leq 0.5 \\ X_1^* = (0,0), X_2^* = (10,0) & \text{if } \alpha \geq 0.5 \end{cases}$$

as an optimal solution. In fact, for all increasing distribution rules we can choose any point in the rectangle and put both new facilities on this point since any point in the rectangle is a solution to (1-med). For all decreasing distribution rules it is optimal to choose one point on the left edge of the rectangle and one point on its right edge. For the uniform distribution rule, both solutions, $X_1^* = (10,0), X_2^* = (10,0)$ and $X_1^* = (0,0), X_2^* = (10,0)$ are optimal.

b) for weights $w_1 = 1$, $w_2 = w_3 = w_4 = 2$ we can compute that

$$\begin{cases} X_1^* = (10,5), X_2^* = (10,5) & \text{if } \alpha \leq \frac{4}{7} \\ X_1^* = (0,5), X_2^* = (10,5) & \text{if } \alpha \geq \frac{4}{7} \end{cases}$$

where in this case the 1-median $(10,5)$ is unique and for $\alpha = \frac{4}{7}$ both solutions are optimal.

c) Finally, for other weights, even if the 1-median is also the unique point $(10,5)$ the result is not exactly the same. E.g., for $w_1 = 1$, $w_2 = 2$, $w_3 = 3$, $w_4 = 2$, we receive

$$\begin{cases} X_1^* = (10,5), X_2^* = (10,5) & \text{if } \alpha \leq \frac{5}{8} \\ X_1^* = (0,5), X_2^* = (10,5) & \text{if } \alpha \geq \frac{5}{8} \end{cases}$$

If we look for $p = 3$ new facilities for the same set of demand points, we get the following picture for unit weights $w_j = 1$, $j \in [4]$:

There are basically two different solutions. In the first solution **Sol1**, all three new facilities are placed in three different vertices of the rectangle. Note that it does not matter which vertices are chosen; due to symmetry all these solutions have the same objective function values for all distribution rules. The second solution **Sol2** places all three new facilities on the same vertex. Clearly, all increasing distribution rules λ_{inc} have **Sol2** as optimal solution, but also many others such as $(0.3, 0.4, 0.3)$, $(0.2, 0, 0.8)$, $(0.3, 0, 0.7)$, $(0.1, 0.8, 0.1)$, or, $(0, 0.8, 0.2)$. On the other hand, **Sol1** is optimal for many decreasing distribution rules, but also for, e.g., $\lambda = (0.8, 0, 0.2), \lambda = (0.2, 0.8, 0)$, or $\lambda = (0.5, 0.1, 0.4)$. We remark that simple computations show that for all symmetric distribution rules $(\alpha, 1-2\alpha, \alpha)$ (e.g., $(0,1,0)$, $(0.2, 0.6, 0.2)$ or $(0.5, 0, 0.5)$) both solutions have the same objective function value.

5.3 A rather symmetric example

We consider a rather symmetric example with seven demand points given as

$$A_1 = (0,0), A_2 = (1,2), A_3 = (2,1), A_4 = (6,6), A_5 = (10,10), A_6 = (11,12), A_7 = (12,11)$$

and unit weights $w_j = 1, j \in [7]$. The (unique) 1-median $X^*_{\text{1-med}} = (6,6)$ is the optimal solution for all increasing distribution rules.

For $p = 2$ some results are depicted in Figure 2. We see that starting from the uniform distribution rule with both new facilities placed on $X^*_{\text{1-med}} = (6,6)$ the two facilities move away from each other when we increase the first weight α. For $\lambda = (1,0)$, the two small squares in the third picture show the possible optimal solutions for the two facilities.

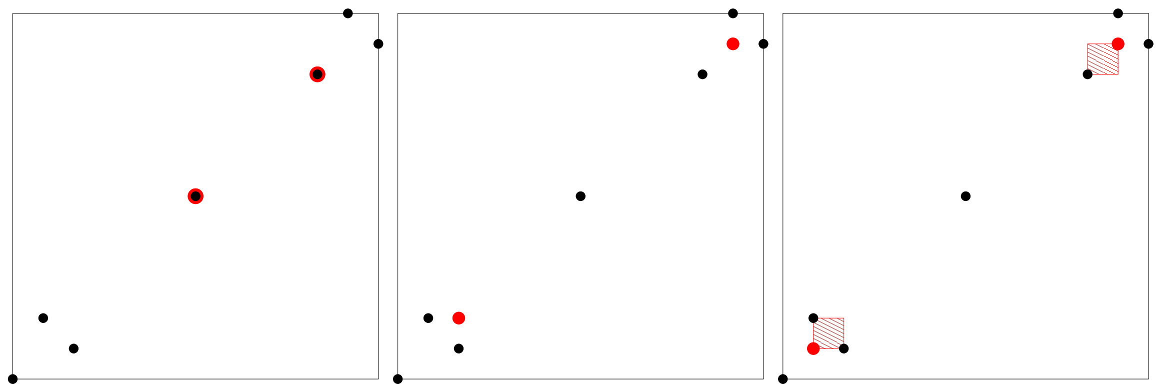

Figure 2: Results for the distribution rules $(0.6, 0.4)$ **(left),** $(0.8, 0.2)$ **(middle), and** $(1, 0)$ **(right).**

For $p = 3$ we depict three interesting instances in Figure 3: The left figure shows the solution of the 3-median problem, i.e., with the distribution rule $(1, 0, 0)$. In the middle we see the solution for the distribution rule $(0, 1, 0)$. Here, two facilities are placed on $X^*_{\text{1-med}}$ to ensure that the second-closest facility is as close as possible to the demand points. The third facility can then be placed anywhere, even very far from *all* demand points. On the right side we finally show the results for the distribution rule $(0.5, 0.3, 0.2)$. This is a rather realistic rule: half of the customers visit the closest facility, 30% the second closest and $\frac{1}{5}$ go to the third-closest facility.

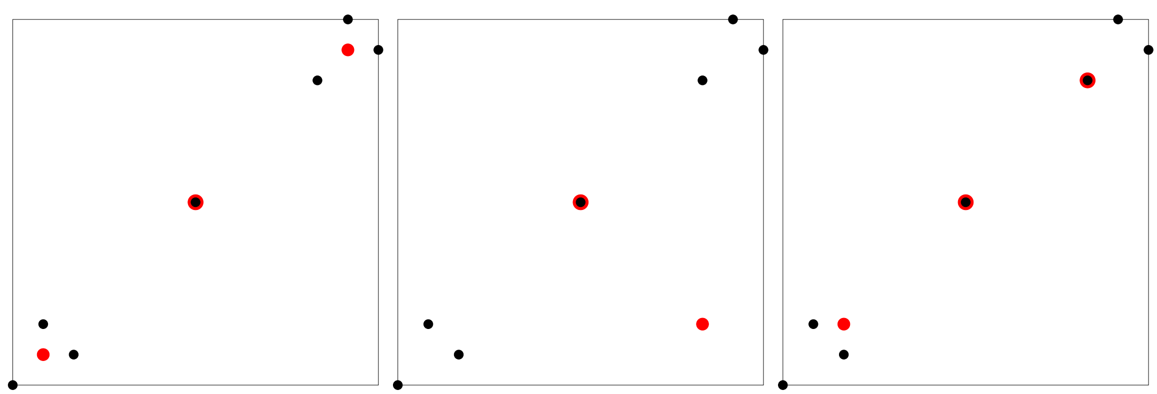

Figure 3: Results for the distribution rules $(1, 0, 0)$ **(left),** $(0, 1, 0)$ **(middle), and** $(0.5, 0.3, 0.2)$ **(right).**

5.4 A random example

For this example we randomly chose 11 demand points with uniform weights on a 10×10 grid with integer coordinates and computed the resulting optimal solutions by the mixed integer linear program of Section 5.1.

The results for $p = 2$ are depicted in Figure 4. The figure shows the results for six different distribution rules $(\alpha, 1 - \alpha)$. The upper left picture shows the result for $\alpha \leq 0.5$, i.e., for all increasing distribution rules. Here both new facilities are placed on the 1-median point. We observe that the two new facilities move apart from each other if we increase α. The situation depicted in the lower right picture is the solution of the classical 2-median problem where each demand point is served by its closest facility.

Finally, we turn to the case of choosing $p = 3$ new facilities in the random instance. Clearly, placing all three new facilities on the 1-median $X^*_{\text{1-med}} = (5, 7)$ is the optimal solution for all increasing distribution rules, and furthermore, also, e.g., for $(0.1, 0.8, 0.1)$, or $(0.1, 0.5, 0.3)$. Figure 5 shows some results for other cases. Note that computation times for $p = 3$ were slightly more than one hour for several of the tested distribution rules.

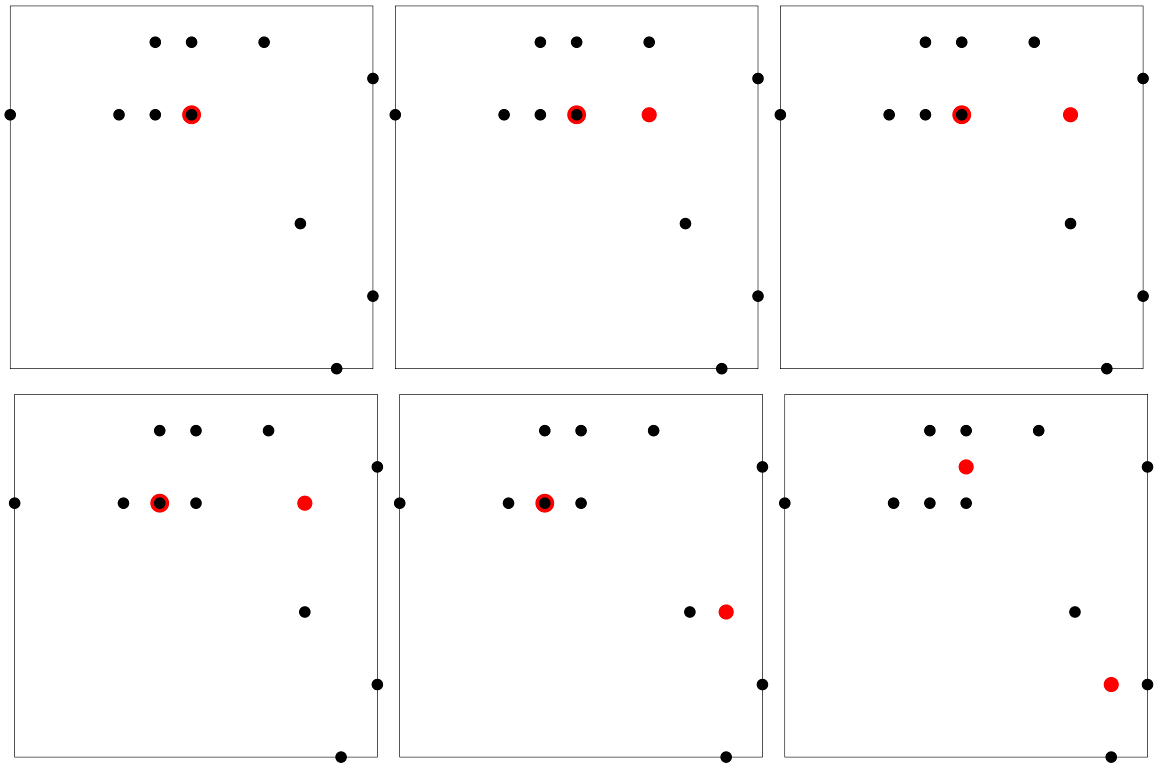

Figure 4: Results for the six distribution rules $(0.5, 0.5)$**,** $(0.6, 0.4)$**,** $(0.7, 0.3)$ **in the upper row and** $(0.8, 0.2)$**,**$(0.9, 0.1)$**, and** $(1, 0)$ **in the lower row.**

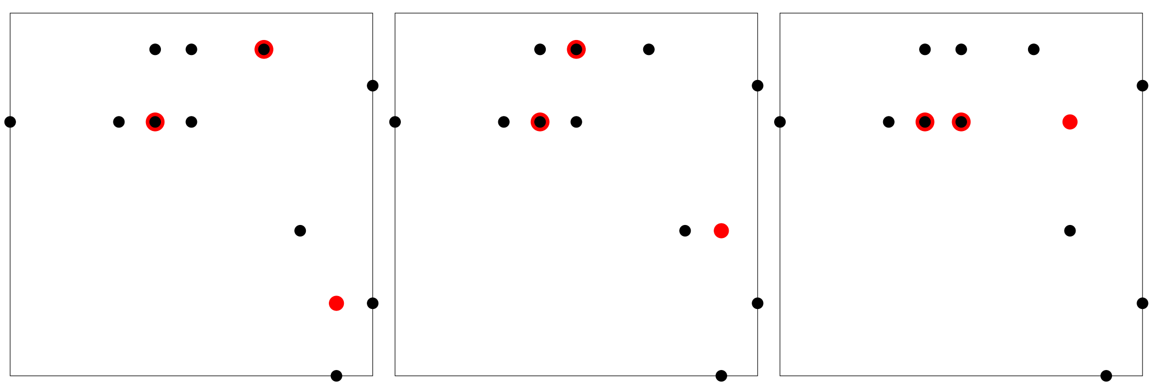

Figure 5: Results for $(1, 0, 0)$ **(left)** $(0.8, 0.1, 0.1)$ **(middle), and** $(0.5, 0.3, 0.2)$ **(right).**

The left picture of Figure 5 shows the solution of the 3-median problem, i.e., the optimal solution with respect to the closest facility distribution rule. Changing the rule only slightly to $\lambda = (0.8, 0.1, 0.1)$ already changes the picture, as shown in the middle. A realistic distribution rule $\lambda = (0.5, 0.3, 0.2)$ is shown in the right picture. Note that it contains the 1-median $X^*_{\text{1-med}} = (5, 7)$, and that the three facilities all lie on the same horizontal line and are rather close together.

6 Conclusion

This paper introduces a new problem which we name the distributed p-median problem where customers do not necessarily visit their closest facilities all the time, but instead, split their demands among several facilities according to some given distribution rule. In this context the classical p-median problem becomes a special case of the generalized model. We investigate increasing and decreasing distribution rules and derive structural properties for both cases and for both, continuous and discrete versions of the mode. We show that the case of an increasing distribution rule can be reduced to a simple 1-median problem. For the (more important) case of decreasing distribution rules we develop nonlinear and integer programming formulations which are considerable simplifications of the general formulations presented, and which prove that each of the facilities is an optimal solution of an adapted 1-median problem.

Further research will entail the design of algorithms for solving distributed p-median problems. These may include Cooper-type heuristics for decreasing distribution rules, improved integer programming formulations (e.g., breaking symmetry in the resulting programs), and identification of finite dominating sets for special distances and distribution rules. Also the investigation of other categories of distribution rules, e.g., symmetric rules, or rules containing constant and variable parts, may be interesting. Finally, instead of median-type location problems, one may also consider distributed center location problems, or one may locate lines, circles, or other structures within a distributed setting.

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